



A T M E
College of Engineering



**MATHEMATICS-II FOR COMPUTER
ENGINEERING STREAMS
(BMATS201)**

ATME COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS

Course objectives:

- The goal of the course **Mathematics-II for Computer science engineering streams(BMATS201)** is to
- **Familiarize** the importance of Integral calculus and Vector calculus essential for Mechanical engineering.
- **Analyze** Mechanical engineering problems applying Partial Differential Equations.
- **Develop** the knowledge of solving Mechanical engineering problems numerically

Course outcome (Course Skill Set)

After successfully completing the course, the student will have a good understanding of the following topics and their applications:

- CO1 Apply the concept of change of order of integration and variables to evaluate multiple integral and their usage in computing area and volume.
- CO2 Understand the applications of vector calculus refer to solenoidal, irrotational vectors, orthogonal curvilinear coordinates.
- CO3 Demonstrate the idea of Linear dependence and independence of sets in the vector space, and linear transformation
- **CO4:** Apply the knowledge of numerical methods in analysing the discrete data and for solving the physical and engineering problems.
- **CO5:** Get familiarize with modern mathematical tools namely SCILAB/PYTHON/MATLAB



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CONTENTS



- Module-1: Integral Calculus
- Module-2: Vector Calculus
- Module-3: Vector Space and Linear Transformations
- Module-4: Numerical methods -1
- Module-5: Numerical methods -2

Module-1: Integral Calculus

- Multiple Integrals: Evaluation of double and triple integrals
- Evaluation of double integrals by change of order of integration, changing into polar coordinates.
- Applications to find Area and Volume by a double integral-Problems.
- Beta and Gamma functions: Definitions, properties, the relation between Beta and Gamma functions.
- Problems on Beta and Gamma functions.

Self-study: Centre of gravity.

Multiple Integral

Double Integral

$$\iint f(x, y) dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dy dx$$

Evaluate the following:

$$1. \int_0^1 \int_x^{\sqrt{x}} xy \, dy dx$$

$$\text{Sol: } I = \int_0^1 x \left[\frac{y^2}{2} \right]_x^{\sqrt{x}} dx = \frac{1}{2} \int_0^1 x [x - x^2] dx$$

$$= \frac{1}{2} \int_0^1 [x^2 - x^3] dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{24}$$

$$2. \int_0^1 \int_0^y xy(x+y)dydx$$

$$\text{Sol: } I = \int_0^1 \int_0^y (x^2y + y^2x)dydx$$

$$= \int_0^1 \left[y \frac{x^3}{3} + y^2 \frac{x^2}{2} \right]_0^y dy$$

$$= \int_0^1 \left[\frac{y^4}{3} + \frac{y^4}{2} \right] dy$$

$$= \int_0^1 \frac{5y^4}{6} dy$$

$$= \frac{5}{6} \left[\frac{y^5}{5} \right]_0^1$$

$$I = \frac{1}{6}$$

Triple integrals:

$$\iiint f(x, y, z) dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z) dz dy dx$$

Evaluate the following:

1. $\int_0^1 \int_0^x \int_0^{xz} xyz \, dz dy dx$

Sol: Let $I = \int_0^1 \int_0^x \int_0^{xz} xyz \, dz dy dx$

$$= \int_0^1 \int_0^x xz \left[\frac{y^2}{2} \right]_0^{xz} dz dx$$
$$= \frac{1}{2} \int_0^1 \int_0^x xz (x^2 z^2 - 0) dz dx$$
$$= \frac{1}{2} \int_0^1 \int_0^x x^3 z^3 dz dx$$

$$\begin{aligned} I &= \frac{1}{2} \int_0^1 x^3 \left[\frac{z^4}{4} \right]_0^x dx \\ &= \frac{1}{8} \int_0^1 x^3 (x^4 - 0) dx \\ &= \frac{1}{8} \int_0^1 x^7 dx \\ &= \frac{1}{8} \left[\frac{x^8}{8} \right]_0^1 = \frac{1}{64} \\ \therefore I &= \frac{1}{64} \end{aligned}$$

Evaluation of double integrals- change of order of integration

1. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by changing the order of the integration.

Sol: Let $I = \int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$

Given Limits $x : x = 0 \text{ to } x = 1$

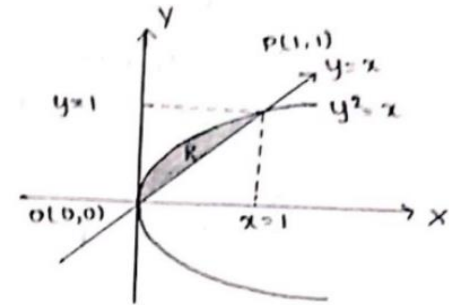
$y : y = x \text{ to } y = \sqrt{x} \text{ or } y^2 = x$

On changing the order of the integration.

$x : x = y^2 \text{ to } x = y$

$y : y = 0 \text{ to } y = 1$

$\therefore I = \int_{y=0}^1 \int_{x=y^2}^y y \, (x \, dx) \, dy$



$$\begin{aligned} I &= \int_{y=0}^1 y \left[\frac{x^2}{2} \right]_{y^2}^y dy \\ &= \frac{1}{2} \int_{y=0}^1 y(y^2 - y^4) dy \\ &= \frac{1}{2} \int_{y=0}^1 (y^3 - y^5) dy \\ &= \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1 \\ \therefore I &= \frac{1}{24} \end{aligned}$$

Evaluation by changing into Polar coordinates

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy dx$ by changing to polar coordinates

Sol: In polar we have $x = r \cos \theta$, $y = r \sin \theta$

$\therefore x^2 + y^2 = r^2$ and $dx dy = r dr d\theta$

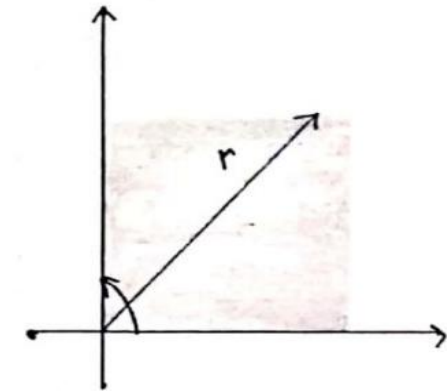
Since x varies from 0 to $\infty \Rightarrow r$ varies from 0 to ∞

In the 1st quadrant θ varies from 0 to $\frac{\pi}{2}$

$$\therefore I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta$$

$$\text{Put } r^2 = t \Rightarrow 2r dr = dt \Rightarrow r dr = \frac{dt}{2}$$

and t also varies from 0 to ∞



$$\begin{aligned} I &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2} d\theta \\ &= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-1} \right]_{t=0}^{\infty} d\theta \\ &= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} -(0 - 1) d\theta \\ &= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} 1 d\theta \\ &= \frac{1}{2} \left[\theta \right]_{\theta=0}^{\frac{\pi}{2}} \\ \therefore I &= \frac{\pi}{4} \end{aligned}$$



Application of Double and Triple integrals

Area of Region R in the Cartesian form: $\iint_R dx dy$

Area of Region R in the Polar form: $\iint_R r dr d\theta$

Volume of solid in the Cartesian form: $\iiint_V dx dy dz$

Volume of solid in the Polar form: $\iint_R 2\pi r^2 \sin\theta dr d\theta$

1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

Sol: Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is ellipse

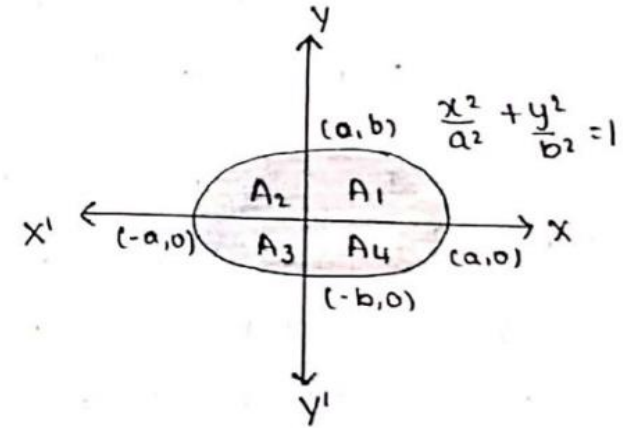
$$\text{Area, } A = 4 \int_{x=0}^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy dx$$

$$= 4 \int_{x=0}^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{x=0}^a \frac{b}{a} \sqrt{a^2-x^2} dx$$

$$= \frac{4b}{a} \int_{x=0}^a \sqrt{a^2-x^2} dx$$

$$= \frac{4b}{a} \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right]_0^a$$



$$A = \frac{4b}{a} \left[\frac{a^2}{2} \sin^{-1}(1) \right]$$

$$= \frac{4b}{a} \left[\frac{a^2}{2} \frac{\pi}{2} \right]$$

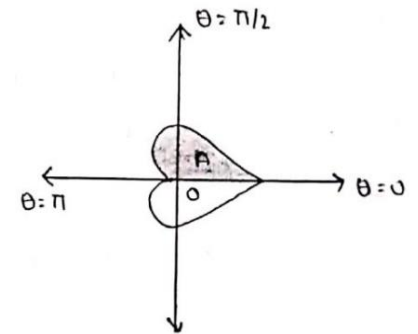
$$\therefore A = \pi ab \text{ sq. units}$$

2. Find the volume generated by the revolution of the cardioid

$r = a(1 + \cos\theta)$ about the initial line using double integral.

Sol: Wkt , $V = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$

$$V = \int_{\theta=0}^{\pi} 2\pi \sin\theta \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} \, d\theta$$



$$\begin{aligned} V &= \int_{\theta=0}^{\pi} 2\pi \sin\theta \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} d\theta \\ &= \frac{2\pi}{3} \int_{\theta=0}^{\pi} \sin\theta a^3 (1 + \cos\theta)^3 d\theta \\ &= \frac{2a^3\pi}{3} \int_{\theta=0}^{\pi} \sin\theta (1 + \cos\theta)^3 d\theta \end{aligned}$$

*Put $1 + \cos\theta = t \Rightarrow -\sin\theta d\theta = dt \Rightarrow \sin\theta d\theta = -dt$
if θ varies from 0 to π then t also varies from 2 to 0*

$$V = \frac{2a^3\pi}{3} \int_{t=2}^0 (t)^3 \cdot -dt = \frac{-2a^3\pi}{3} \left[\frac{t^4}{4} \right]_2^0$$

$$V = \frac{-a^3 \pi}{6} (0 - 16)$$

$$\therefore V = \frac{8a^3 \pi}{3} \text{ cubic units}$$

Beta and Gamma functions:

Definition:

Let n be any positive number. The *gamma* function of n is denoted by $\Gamma(n)$ and it is defined as

$$\Gamma(n) = \int_{x=0}^{\infty} e^{-x} x^{n-1} dx$$

The *beta* function is denoted by $\beta(m, n)$ and it is defined as

$$\beta(m, n) = \int_{x=0}^1 x^{m-1} (1-x)^{n-1} dx, \quad m, n > 0.$$

Relation between Beta and Gamma functions

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Note:

1. $\beta(m, n) = 2 \int_{x=0}^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$

2. $\Gamma(n) = \int_{x=0}^{\infty} e^{-t^2} t^{2n-1} dt$

Properties of Beta and Gamma function

1. $\beta(m, n) = \beta(n, m)$
2. $\Gamma(n + 1) = n\Gamma(n)$
3. $\Gamma(n + 1) = n!$, for any positive integer n
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
5. $\int_{x=0}^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$
6. $\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

Module-2: Vector Calculus

- Vector Differentiation: Scalar and vector fields. Gradient, directional derivative.
- curl and divergence - physical interpretation, solenoidal and irrotational vector fields. Problems.
- Vector Integration: Line integrals, Surface integrals.
- Applications to work done by a force and flux.
- Statement of Green's theorem and Stoke's theorem-Problems.

Self-study: Volume integral and Gauss divergence theorem.

VECTOR CALCULUS

Vector calculus is a field of mathematics concerned with multivariate real analysis of vectors in two or more dimensions. It consists of set of problems solving techniques very useful for engineering and physics.

SCALAR AND VECTOR POINT FUNCTIONS

Let $\vec{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ be a 'Vector function'. Then for various values of t we get a set of constant vectors.

Let $\varphi = \varphi(x, y, z)$ be a 'Scalar function'. Then for various values of x, y, z we get a set of points or scalars.

Vector operation ∇ (*del*) is defined by the equation

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

This operator has a great role in vector calculus. Laplacian operator ∇^2 is defined as follows

$$\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

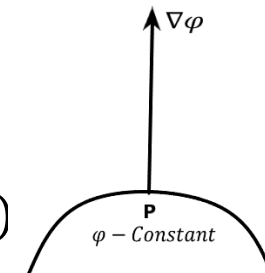
Gradient

The vector function $\nabla\varphi$ is defined as the gradient of the scalar function $\varphi = \varphi(x, y, z)$

i. e., $grad\varphi = \nabla\varphi = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \varphi$

$$grad\varphi = \nabla\varphi = \left(\frac{\partial\varphi}{\partial x} \hat{i} + \frac{\partial\varphi}{\partial y} \hat{j} + \frac{\partial\varphi}{\partial z} \hat{k} \right)$$

Geometrically, $\nabla\varphi$ represents a normal at any point P to the surface $\varphi(x, y, z) = constant$ and has a magnitude equal to the rate of change of $\varphi(x, y, z)$ along this normal. $\nabla\varphi$ is a **vector quantity**.



Note:

1. The unit normal vector \hat{n} along $\nabla\varphi$ is given by $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$ or $\hat{n} = \frac{\nabla f}{|\nabla f|}$
2. The component of $\nabla\varphi$ in the direction of a unit vector \vec{a} is $\nabla\varphi \cdot \hat{n}$ and is called the *directional derivative* of φ in the direction of \vec{a} . Thus, the directional derivative is maximum in the direction $\nabla\varphi$ and the magnitude of this maximum is equal to $|\nabla\varphi|$.
i. e., $D_{\vec{a}}\varphi = \nabla\varphi \cdot \hat{n}$ where $\hat{n} = \frac{\vec{a}}{|\vec{a}|}$

Problems

1. Find the unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$.

Sol: Let $\varphi = x^3 + y^3 + 3xyz - 3$

$$\frac{\partial\varphi}{\partial x} = 3x^2 + 3yz$$

$$\frac{\partial\varphi}{\partial y} = 3y^2 + 3xz$$

$$\frac{\partial\varphi}{\partial z} = 3xy$$

$$\text{Now, } \nabla\varphi = \left(\frac{\partial\varphi}{\partial x} \hat{i} + \frac{\partial\varphi}{\partial y} \hat{j} + \frac{\partial\varphi}{\partial z} \hat{k} \right)$$

$$\nabla\varphi = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

At (1,2,-1)

$$\nabla\varphi = (3 - 6)\hat{i} + (12 - 3)\hat{j} + (6)\hat{k}$$

$$\nabla\varphi = -3\hat{i} + 9\hat{j} + 6\hat{k}$$

$$|\nabla\varphi| = \sqrt{(-3)^2 + (9)^2 + (6)^2} = \sqrt{9 + 81 + 36}$$

$$|\nabla\varphi| = \sqrt{126}$$

$$\text{The unit normal vector, } \hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{126}}$$

2. Find the directional derivative of the function $f = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ along $2\hat{i} - 3\hat{j} + 6\hat{k}$

Sol: Given $f = 4xz^3 - 3x^2y^2z$ Let $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$$\frac{\partial f}{\partial x} = 4z^3 \cdot 1 - 3y^2z \cdot 2x$$

$$\frac{\partial f}{\partial y} = 0 - 3x^2z \cdot 2y$$

$$\frac{\partial f}{\partial z} = 4x \cdot 3z^2 - 3x^2y^2 \cdot 1$$

$$\frac{\partial f}{\partial x} = 4z^3 - 6xy^2z$$

$$\frac{\partial f}{\partial y} = -6x^2yz$$

$$\frac{\partial f}{\partial z} = 12xz^2 - 3x^2y^2$$

$$\text{Now, } \nabla f = \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right)$$

$$\nabla f = (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

At $(2, -1, 2)$

$$\nabla f = (32 - 24)\hat{i} + (48)\hat{j} + (96 - 12)\hat{k}$$

$$\nabla f = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

$$\text{Also, } \vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + (-3)^2 + 6^2}$$

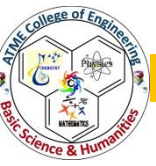
$$|\vec{a}| = \sqrt{4 + 9 + 36}$$

$$|\vec{a}| = \sqrt{49}$$

$$|\vec{a}| = 7$$

$$\therefore \hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

$$D.D = \nabla f \cdot \hat{n}$$



$$D \cdot D = (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{(2\hat{i} - 3\hat{j} + 6\hat{k})}{7}$$

$$D \cdot D = \frac{(8)(2) + (48)(-3) + (84)(6)}{7}$$

$$D \cdot D = \frac{16 - 144 + 504}{7}$$

$$D \cdot D = \frac{376}{7}$$

3. Find the angle between the surfaces or normal surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.

Sol: Given $\varphi = x^2 + y^2 + z^2 - 9$ $\psi = x^2 + y^2 - z - 3$

$$\frac{\partial \varphi}{\partial x} = 2x$$

$$\frac{\partial \psi}{\partial x} = 2x$$

$$\frac{\partial \varphi}{\partial y} = 2y$$

$$\frac{\partial \psi}{\partial y} = 2y$$

$$\frac{\partial \varphi}{\partial z} = 2z$$

$$\frac{\partial \psi}{\partial z} = -1$$

Wkt, $\nabla \varphi = \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k}$

$$\nabla \varphi = (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$$

At $(2, -1, 2)$

$$\nabla\varphi = (2.2)\hat{i} + (2.(-1))\hat{j} + (2.2)\hat{k}$$

$$\nabla\varphi = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$|\nabla\varphi| = \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36}$$

$$|\nabla\varphi| = 6$$

$$\text{Also } \nabla\psi = \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$$

$$\nabla\psi = (2x)\hat{i} + (2y)\hat{j} + (-1)\hat{k}$$

At $(2, -1, 2)$

$$\nabla\psi = (2.2)\hat{i} + (2.(-1))\hat{j} + (-1)\hat{k}$$

$$\nabla\psi = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$|\nabla\psi| = \sqrt{4^2 + (-2)^2 + (-1)^2}$$

$$|\nabla\psi| = \sqrt{21}$$

Angle between surface,

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$$

$$\cos\theta = \frac{\nabla\phi}{|\nabla\phi|} \cdot \frac{\nabla\psi}{|\nabla\psi|}$$

$$\cos\theta = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k})}{6} \cdot \frac{(4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{21}}$$

$$\cos\theta = \frac{16 + 4 - 4}{8 \cdot 6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence of a vector function

The divergence of a vector function $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, where f_1, f_2, f_3 are functions of x, y, z . It is denoted by $div\vec{F}$ and is defined as

$$div\vec{F} = \nabla \cdot \vec{F}$$

$$div\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

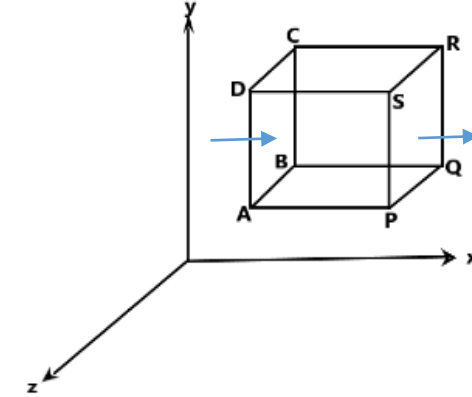
Clearly $div\vec{F}$ is a scalar quantity.

Physical interpretation of Divergence

Let us consider the motion of the fluid. Consider a small rectangular parallelepiped with edges $\delta_x, \delta_y, \delta_z$ parallel to the axes in the mass of fluid.

Let $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$ be the velocity of the fluid at (x, y, z) .

Amount of the fluid flowing in through the face ABCD per unit time
 = Velocity \times Area of the face
 = $V_x \delta y \delta z$



Amount of the fluid flowing out through the face PQRS per unit time
 = $\left[V_x + \frac{\partial V_x}{\partial x} \delta x \right] \delta y \delta z$

\therefore The net decrease in the amount of fluid across these two faces is

$$= \left[V_x + \frac{\partial V_x}{\partial x} \delta x \right] \delta y \delta z - V_x \delta y \delta z$$

$$= \left[V_x + \frac{\partial V_x}{\partial x} \delta x - V_x \right] \delta y \delta z = \frac{\partial V_x}{\partial x} \delta x \delta y \delta z$$

Similarly, the decrease in amount of fluid due to flow along the $y - \text{axis} = \frac{\partial V_y}{\partial y} \delta x \delta y \delta z$

The decrease in amount of fluid due to flow along the $z - \text{axis} = \frac{\partial V_z}{\partial z} \delta x \delta y \delta z$

Total decrease in amount of fluid inside the parallelepiped per unit time
$$= \left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right] \delta x \delta y \delta z$$

Hence the ratio of loss of fluid per unit volume $= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$
$$= \left[\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right] \cdot [V_x \hat{i} + V_y \hat{j} + V_z \hat{k}]$$

$$= \nabla \cdot \vec{V}$$

$$= \text{div} \vec{V}$$

Hence $div\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid. If $div\vec{V} = 0$ everywhere in some region of space, then \vec{V} is called **Solenoidal Vector function** and the fluids said to be *incompressible* i.e., there is no gain or loss in the volume element

Curl of a vector function

The curl of a vector function $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is denoted by $curl\vec{F}$ and is defined as $curl\vec{F} = \nabla \times \vec{F}$

$$curl\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$curl\vec{F} = \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right] \hat{i} - \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right] \hat{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right] \hat{k}$$

Clearly $curl\vec{F}$ is a vector quantity.

Physical interpretation of Curl

Consider a rigid body rotating about a fixed axis through origin. Let the uniform angular velocity be $\vec{\omega} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$, w_1, w_2, w_3 are constants. The velocity \vec{V} of any point P(x, y, z) on the body is given by $\vec{V} = \vec{\omega} \times \vec{r}$, where \vec{r} is the position vector of P.

Let $\vec{\omega} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Consider $\vec{V} = \vec{\omega} \times \vec{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$\vec{V} = (w_2z - w_3y)\hat{i} - (w_1z - w_3x)\hat{j} + (w_1y - w_2x)\hat{k}$$

$$\text{curl} \vec{V} = \nabla \times \vec{V}$$

$$\text{curl} \vec{V} = \nabla \times [(w_2z - w_3y)\hat{i} - (w_1z - w_3x)\hat{j} + (w_1y - w_2x)\hat{k}]$$

$$\text{curl} \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (w_2z - w_3y) & (w_3x - w_1z) & (w_1y - w_2x) \end{vmatrix}$$

$$\text{curl} \vec{V} = [w_1 - (-w_1)]\hat{i} - [-w_2 - w_2]\hat{j} + [w_3 - (-w_3)]\hat{k}$$

$$\text{curl} \vec{V} = [2w_1]\hat{i} + [2w_2]\hat{j} + [2w_3]\hat{k}$$

$$\text{curl} \vec{V} = 2[w_1\hat{i} + w_2\hat{j} + w_3\hat{k}]$$

$$\text{curl} \vec{V} = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2} \text{curl} \vec{V}$$

Thus the angular velocity of rotation at any point is equal to half of the curl of the velocity.

Note:

1. If $\text{div} \vec{F} = \mathbf{0}$, then we say that \vec{F} is **Solenoidal** vector.
2. If $\text{curl} \vec{F} = \mathbf{0}$, then we say that \vec{F} is **irrotational** vector.
3. Irrotational vector field is called as conservative field or potential field.
4. When \vec{F} is irrotational there always exist a scalar point function such that $\nabla \phi = \vec{F}$, then ϕ is called a scalar potential of vector \vec{F} .

Problems

1. If $\vec{F} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$, find the $div\vec{F}$ and $curl\vec{F}$ at $(2, -1, 1)$.

Sol: Given $\vec{F} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$

Wkt $div\vec{F} = \nabla \cdot \vec{F}$

$$div\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$div\vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$div\vec{F} = yz \cdot 1 + 3x^2 \cdot 1 + (x \cdot 2z - y^2 \cdot 1)$$

$$div\vec{F} = yz + 3x^2 + 2xz - y^2$$

At $(2, -1, 1)$

$$div\vec{F} = (-1)(1) + 3 \cdot (2)^2 + 2(2)(1) - (-1)^2$$

$$div\vec{F} = -1 + 12 + 4 - 1$$

$$div\vec{F} = 14$$

$$\begin{aligned}\text{Also, } \text{curl} \vec{F} &= \nabla \times \vec{F} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xyz) & (3x^2y) & (xz^2 - y^2z) \end{vmatrix}\end{aligned}$$

$$\text{curl} \vec{F} = \left[\frac{\partial}{\partial y} (xz^2 - y^2z) - \frac{\partial}{\partial z} (3x^2y) \right] \hat{i} - \left[\frac{\partial}{\partial x} (xz^2 - y^2z) - \frac{\partial}{\partial z} (xyz) \right] \hat{j} + \left[\frac{\partial}{\partial x} (3x^2y) - \frac{\partial}{\partial y} (xyz) \right] \hat{k}$$

$$\text{curl} \vec{F} = [(-2yz) - 0] \hat{i} - [(z^2 - 0) - (xy)] \hat{j} + [(6xy) - (xz)] \hat{k}$$

$$\text{curl} \vec{F} = [-2yz] \hat{i} - [z^2 - xy] \hat{j} + [6xy - xz] \hat{k}$$

At (2,-1,1)

$$\text{curl} \vec{F} = [-2(-1)(1)] \hat{i} - [1^2 - 2(-1)] \hat{j} + [6(2)(-1) - (2)(1)] \hat{k}$$

$$\text{curl} \vec{F} = 2\hat{i} - 3\hat{j} - 14\hat{k}$$

2. Find $\text{div}\vec{F}$ and $\text{curl}\vec{F}$ where $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$.

Sol: Let $\varphi = x^3 + y^3 + z^3 - 3xyz$

$$\vec{F} = \text{grad}\varphi$$

$$\vec{F} = \nabla\varphi$$

$$\vec{F} = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$\vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

Now, $\text{div}\vec{F} = \nabla \cdot \vec{F}$

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$\text{div}\vec{F} = (6x - 0) + (6y - 0) + (6z - 0)$$

$$\text{div}\vec{F} = 6x + 6y + 6z$$

$$\text{curl} \vec{F} = \nabla \times \vec{F}$$

$$\text{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$$

$$\text{curl} \vec{F} = \left[\frac{\partial}{\partial y} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3y^2 - 3xz) \right] \hat{i} - \left[\frac{\partial}{\partial x} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3x^2 - 3yz) \right] \hat{j} + \left[\frac{\partial}{\partial x} (3y^2 - 3xz) - \frac{\partial}{\partial y} (3x^2 - 3yz) \right] \hat{k}$$

$$\text{curl} \vec{F} = [(0 - 3x) - (0 - 3x)]\hat{i} - [(0 - 3y) - (0 - 3y)]\hat{j} + [(0 - 3z) - (0 - 3z)]\hat{k}$$

$$\text{curl} \vec{F} = [-3x + 3x]\hat{i} - [-3y + 3y]\hat{j} + [-3z + 3z]\hat{k}$$

$$\text{curl} \vec{F} = 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$\text{curl} \vec{F} = \vec{0}$$

3. Show that $\vec{F} = \frac{x\hat{i}+y\hat{j}}{(x^2+y^2)}$ is both solenoidal and irrotational.

Sol: Given $\vec{F} = \frac{x}{(x^2+y^2)}\hat{i} + \frac{y}{(x^2+y^2)}\hat{j}$

Wkt, $div\vec{F} = \nabla \cdot \vec{F}$

$$div\vec{F} = \frac{\partial}{\partial x} \left(\frac{x}{(x^2+y^2)} \right) + \frac{\partial}{\partial y} \left(\frac{y}{(x^2+y^2)} \right)$$

$$div\vec{F} = \frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2} + \frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}$$

$$div\vec{F} = \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$div\vec{F} = 0$$

$\therefore \vec{F}$ is Solenoidal.

Now, $\text{curl} \vec{F} = \nabla \times \vec{F}$

$$\text{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix}$$

$$\text{curl} \vec{F} = \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z} \left(\frac{y}{x^2 + y^2} \right) \right] \hat{i} - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z} \left(\frac{x}{x^2 + y^2} \right) \right] \hat{j} + \left[\frac{\partial}{\partial x} \left(\frac{y}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) \right] \hat{k}$$

$$\text{curl} \vec{F} = [(0) - (0)] \hat{i} - [(0) - (0)] \hat{j} + \left[\frac{-2xy}{(x^2 + y^2)} + \frac{2xy}{(x^2 + y^2)} \right] \hat{k}$$

$$\text{curl} \vec{F} = 0 \hat{i} - 0 \hat{j} + 0 \hat{k}$$

$$\text{curl} \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

4. P.T $\vec{F} = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$ is irrotational. Also find a scalar point function φ such that $\vec{F} = \nabla\varphi$.

Sol: Given $\vec{F} = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$
 $\text{curl}\vec{F} = \nabla \times \vec{F}$

$$\text{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y + z) & (z + x) & (x + y) \end{vmatrix}$$

$$\text{curl}\vec{F} = \left[\frac{\partial}{\partial y}(x + y) - \frac{\partial}{\partial z}(z + x) \right] \hat{i} - \left[\frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial z}(y + z) \right] \hat{j} + \left[\frac{\partial}{\partial x}(z + x) - \frac{\partial}{\partial y}(y + z) \right] \hat{k}$$

$$\text{curl}\vec{F} = [(1) - (1)]\hat{i} - [(1) - (1)]\hat{j} + [(1) - (1)]\hat{k}$$

$$\text{curl}\vec{F} = 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$\text{curl} \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

To find φ

Consider $\nabla \varphi = \vec{F}$

$$\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} = (y + z) \hat{i} + (z + x) \hat{j} + (x + y) \hat{k}$$

$$\frac{\partial \varphi}{\partial x} = y + z \qquad \frac{\partial \varphi}{\partial y} = z + x \qquad \frac{\partial \varphi}{\partial z} = x + y$$

Integrating we get

$$\varphi = (y + z) \int 1 dx$$

$$\varphi = (z + x) \int 1 dy$$

$$\varphi = (x + y) \int 1 dz$$

$$\varphi = (y + z)x + f(y, z)$$

$$\varphi = (z + x)y + f(x, z)$$

$$\varphi = (x + y)z + f(x, y)$$

$$\varphi = xy + xz + f(y, z)$$

$$\varphi = yz + xy + f(x, z)$$

$$\varphi = xz + yz + f(x, y)$$

$$\text{curl} \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

To find φ

Consider $\nabla \varphi = \vec{F}$

$$\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$$

$$\frac{\partial \varphi}{\partial x} = y + z \qquad \frac{\partial \varphi}{\partial y} = z + x \qquad \frac{\partial \varphi}{\partial z} = x + y$$

Integrating we get

$$\varphi = (y + z) \int 1 dx$$

$$\varphi = (z + x) \int 1 dy$$

$$\varphi = (x + y) \int 1 dz$$

$$\varphi = (y + z)x + f(y, z)$$

$$\varphi = (z + x)y + f(x, z)$$

$$\varphi = (x + y)z + f(x, y)$$

$$\varphi = xy + xz + f(y, z)$$

$$\varphi = yz + xy + f(x, z)$$

$$\varphi = xz + yz + f(x, y)$$

$$\text{curl} \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

To find φ

Consider $\nabla \varphi = \vec{F}$

$$\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} = (y + z) \hat{i} + (z + x) \hat{j} + (x + y) \hat{k}$$

$$\frac{\partial \varphi}{\partial x} = y + z \qquad \frac{\partial \varphi}{\partial y} = z + x \qquad \frac{\partial \varphi}{\partial z} = x + y$$

Integrating we get

$$\varphi = (y + z) \int 1 dx$$

$$\varphi = (z + x) \int 1 dy$$

$$\varphi = (x + y) \int 1 dz$$

$$\varphi = (y + z)x + f(y, z)$$

$$\varphi = (z + x)y + f(x, z)$$

$$\varphi = (x + y)z + f(x, y)$$

$$\varphi = xy + xz + f(y, z)$$

$$\varphi = yz + xy + f(x, z)$$

$$\varphi = xz + yz + f(x, y)$$

$\therefore \varphi = xy + xz + yz$, where $f(y, z) = yz$, $f(x, z) = xz$, $f(x, y) = xy$



Module-3 Vector Space and Linear Transformation

- What is a vector? Many are familiar with the concept of a vector as:
- Something which has magnitude and direction.
- an ordered pair or triple.
- a description for quantities such as Force, velocity and acceleration.
- Such vectors belong to the foundation vector space $-R^n$ - of all vector spaces. The properties of general vector spaces are based on the properties of R^n . It is therefore helpful to consider briefly the nature of R^n

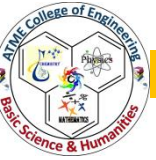


Definitions

- Two vectors $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ in R_n are called equal if $u_1 = v_1, u_2 = v_2, \dots, u_n = v_n$
- The sum $u + v$ is defined by $u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$

Let k be any scalar, then the scalar multiple ku is defined by

- $ku = (ku_1, ku_2, \dots, ku_n)$
- These two operations of addition and scalar multiplication are called the standard operations on R_n .
- The zero vector in R_n is denoted by 0 and is defined to be the vector $0 = (0, 0, \dots, 0)$



- The negative (or additive inverse) of u is denoted by $-u$ and is defined by
$$-u = (-u_1, -u_2, \dots, -u_n)$$
- The difference of vectors in R^n is defined by $v - u = v + (-u)$
- The most important arithmetic properties of addition and scalar multiplication of vectors in R^n are listed in the following theorem. This theorem enables us to manipulate vectors in R^n without expressing the vectors in terms of components.

• Linear Combinations of Vectors and Systems of Linear Equations

Have m linear equations in n variables:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Write in matrix form: $Ax = b$.

$A = [a_{ij}]$ is the $m \times n$ coefficient matrix.

- **Linear combinations Definitions**

A vector w is called a linear combination of the vectors v_1, v_2, \dots, v_r if it can be expressed in the form $w = k_1v_1 + k_2v_2 + \dots + k_rv_r$

where k_1, k_2, \dots, k_r are scalars.

- Example : Consider the vectors $u = (1, 2, -1)$ and $v = (6, 4, 2)$ in R^3 . Show that $w = (9, 2, 7)$ is a linear combination of u and v and that $w_0 = (4, -1, 8)$ is not a linear combination of u and v .

- **Spanning**

If $S = \{v_1, v_2, \dots, v_r\}$ is a set of vectors in a vector space V , then the subspace W of V consisting of all linear combinations of the vectors in S is called the space spanned by v_1, v_2, \dots, v_r , and it is said that the vectors v_1, v_2, \dots, v_r span W . To indicate that W is the space spanned by the vectors in the set $S = \{v_1, v_2, \dots, v_r\}$ the below notation is used.

$W = \text{span}(S)$ or $W = \text{span}\{v_1, v_2, \dots, v_r\}$

Linear transformations: A linear transformation $T : V1 \rightarrow V2$ between two vector spaces is a function preserving all of the algebra; that is, $T(\alpha v + \beta u) = \alpha T(v) + \beta T(u)$ for all scalars $\alpha, \beta \in \mathbb{R}$ and vectors $v, u \in V1$. $V1$ is called the domain of T and $V2$ is called the codomain of T .

A linear transformation $T : V1 \rightarrow V2$ is onto if for each $u \in V2$ there is at least one $v \in V1$ such that $u = T(v)$. Formal differentiation is an onto linear transformation.

A linear transformation $T : V1 \rightarrow V2$ is an isomorphism if it is one to one and onto. When T is an isomorphism, we say $V1$ and $V2$ are isomorphic vector spaces and we write $V1 \cong V2$.

If $T : V1 \rightarrow V2$ is an isomorphism then for any $u \in V2$ there is exactly one $v \in V1$ such that $u = T(v)$. Thus, we can define a function $T^{-1} : V2 \rightarrow V1$ by taking $T^{-1}(u)$ to be the unique $v \in V1$ such that $u = T(v)$. T^{-1} is linear whenever T is linear and T^{-1} is an isomorphism of vector spaces whenever T is an isomorphism.

- Null space.

(a) The Kernel or Null Space of a linear transformation $T : V_1 \rightarrow V_2$ is $\text{Nul}(T) = \{u \mid T(u) = 0\}$

(b) $\text{Nul}(T)$ is a subspace of V_1 .

(c) When T is a matrix transformation, row reducing the augmented matrix $[T \mid 0]$ to reduced echelon form produces a vector equation describing $\text{Nul}(T)$. We call this equation the kernel equation of the matrix. It provides us with a basis of $\text{Nul}(T)$ (see "basis" below).

- **Linearly Independence** : A subset B of a vector space is linearly independent if any finite linear combination $\alpha_1 b_1 + \dots + \alpha_n b_n = 0$ of elements $b_1, \dots, b_n \in B$ implies $\alpha_1 = 0, \dots, \alpha_n = 0$.

- **Basis**: (a) A basis of a vector space V is a set of linearly independent vectors spanning V .

- (b) $\{e_1, e_2, \dots, e_n\}$ is a basis of \mathbb{R}^n . It is called the standard basis of \mathbb{R}^n .

- (c) The set of vectors $\{1, t, t^2, t^3, \dots, t^n\}$ is a basis of P_n . It is the standard basis of P_n .

- **THEOREM**: If $\{b_1, b_2, \dots, b_n\}$ is a basis of a vector space and if v is any vector in the space then there is exactly one set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of scalars such that $v = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \dots + \alpha_n b_n$

- If any vector not in a basis is added to the basis then the resulting set is not a basis.

- If V has a basis $= \{v_1, v_2\}$ and $T : V \rightarrow V$ is a linear transformation given By $T(v_1) = 2v_1 + 3v_2, T(v_2) = -7v_1 + 8v_2$
- To find $T(-v_1 + 3v_2)$ takes two steps as shown below.

$$\begin{aligned} T(-v_1 + 3v_2) &= -T(v_1) + 3T(v_2) \\ &= -2v_1 - 3v_2 + 3(-7v_1 + 8v_2) \\ &= -23v_1 + 21v_2 \end{aligned}$$

$$\begin{aligned} \text{Hence } T(-v_1 + 3v_2) &= T(-23v_1 + 21v_2) \\ &= -23T(v_1) + 21T(v_2) \\ &= -23(2v_1 + 3v_2) + 21(-7v_1 + 8v_2) \\ &= -193v_1 + 99v_2 \end{aligned}$$

Module-4: Numerical Methods-1

- Solution of polynomial and transcendental equations: Regula-Falsi and Newton-Raphson methods (only formulae).
- Finite differences, Interpolation using Newton's forward and backward difference formulae.
- Newton's divided difference formula and Lagrange's interpolation formula.
- Numerical integration: Simpson's (1/3)rd and (3/8)th rules(without proof)-Problems.

Self-study: Bisection method, Lagrange's inverse Interpolation, Weddle's rule.

Numerical Methods

Numerical method provide various technique to find approximate solution to different problem using simple operation.

Numerical Solution of Polynomial and Transcendental Equations

Given an equation $f(x) = 0$ it is generally not possible to find roots ' x ' such that $f(x)$ becomes zero exactly. We discuss two numerical methods for the solution of algebraic and transcendental equation.

Equation involving algebraic quantity like x, x^2, x^3, \dots are called *algebraic equation*.

Eg: $x^3 - 3x - 4 = 0$, $x^4 + x^3 = 80$

Equation involving non algebraic quantity like $e^x, \log x, \sin x, \tan x, \dots$ are called *transcendental equation*.

Eg: $xe^x - 2 = 0$, $x \log x - 1.2 = 0$, $\tan x = 2x$

Numerical methods are often a repetitive nature. This consist repeated execution of the same process where at each step to result of the previous step is used. This is known as iterative process.

We discuss two numerical methods

1. Regula-Falsi method
2. Newton-Raphson method

Regula-Falsi method (or) Method of False position

Formula:

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Problems

1. Using the method of false position find the real root correct to 3 decimal places of the equation $x^3 + 5x - 11 = 0$.

Sol: Given $f(x) = x^3 + 5x - 11$

$$f(0) = -11$$

$$f(1) = -5 < 0$$

$$f(2) = 7 > 0$$

The root lies between (1,2).

Wkt,

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Step-1: $a = 1$ $b = 2$
 $f(a) = -5$ $f(b) = 7$

$$x_1 = \frac{1 \cdot (7) - 2 \cdot (-5)}{(7) - (-5)}$$

$$x_1 = \frac{7 + 10}{7 + 5}$$

$$x_1 = 1.417$$

$$f(1.417) = -1.070 < 0$$

$$f(2) = 7 > 0$$

The root lies between (1.417,2).

Step-2: $a = 1.417$ $b = 2$
 $f(a) = -1.070$ $f(b) = 7$

$$x_2 = \frac{(1.417)(7) - 2(-1.070)}{(7) - (-1.070)}$$

$$x_2 = \mathbf{1.494}$$

$$f(1.494) = -0.195 < 0$$

$$f(2) = 7 > 0$$

The root lies between (1.417,2).

Step-3: $a = 1.494$ $b = 2$
 $f(a) = -0.195$ $f(b) = 7$

$$x_3 = \frac{(1.494)(7) - 2(-0.195)}{(7) - (-0.195)}$$

$$x_3 = \mathbf{1.508}$$

$$f(1.508) = -0.031 < 0$$

$$f(2) = 7 > 0$$

The root lies between (1.508,2).

Step-4: $a = 1.508$ $b = 2$
 $f(a) = -0.031$ $f(b) = 7$

$$x_4 = \frac{(1.508)(7) - 2(-0.031)}{(7) - (-0.031)}$$

$$x_4 = \mathbf{1.510}$$

$$f(1.510) = -0.007 < 0$$

$$f(2) = 7 > 0$$

The root lies between (1.510,2).

Step-5: $a = 1.510$ $b = 2$
 $f(a) = -0.007$ $f(b) = 7$

$$x_5 = \frac{(1.510)(7) - 2(-0.007)}{(7) - (-0.007)}$$

$$x_5 = \mathbf{1.510}$$

∴ The real root is 1.510

Newton-Raphson Method

Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} ; f'(x_n) \neq 0 , n = 0,1,2,3, \dots$$

$$n = 0 , x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} ; f'(x_0) \neq 0$$

$$n = 1 , x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} ; f'(x_1) \neq 0 \quad \text{and so on.}$$

Problems

1. Use NR method to find the real root of the equation $x^3 - 3x - 5 = 0$, correct to 3 decimal places.

Sol: Given $f(x) = x^3 - 3x - 5$

$$f(0) = -5$$

$$f(1) = -7$$

$$f(2) = -3$$

$$f(3) = 13$$

The root lies between (2,3).

Since $f(2)$ lies nearer to 0.

Let $x_0 = 2$

We know that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Here $f(x) = x^3 - 3x - 5$

$$f'(x) = 3x^2 - 3$$

Step 1: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_1 = 2 - \frac{f(2)}{f'(2)}$$

$$x_1 = 2 - \frac{(-3)}{(9)}$$

$$x_1 = \mathbf{2.333}$$

Step 2: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$x_2 = 2.333 - \frac{f(2.333)}{f'(2.333)}$$

$$x_2 = 2.333 - \frac{(0.699)}{(13.329)}$$

$$x_2 = \mathbf{2.281}$$

Step 3: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$x_3 = 2.281 - \frac{f(2.281)}{f'(2.281)}$$

$$x_3 = 2.281 - \frac{(0.025)}{(12.609)}$$

$$x_3 = \mathbf{2.279}$$

Step 4: $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$x_4 = 2.279 - \frac{f(2.279)}{f'(2.279)}$$

$$x_4 = 2.279 - \frac{(0)}{(12.582)}$$

$$x_4 = \mathbf{2.279}$$

∴ The real root is 2.279.

FINITE DIFFERENCES

Newton's Forward interpolation formula (NFIF)

The value of $y = f(x)$ at $x = x_0 + rh$ is approximately given by

$$y = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots + \frac{r(r-1)(r-2)\dots[r-(n-1)]}{n!}\Delta^n y_0$$

Where, ' r ' is any real number, $r = \frac{x-x_0}{h}$; h is step length.

Newton's Backward interpolation formula (NBIF)

The value of $y = f(x)$ at $x = x_n + rh$ is approximately given by

$$y = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \dots + \frac{r(r+1)(r+2)\dots[r+(n-1)]}{n!}\nabla^n y_n$$

Where, ' r ' is any real number, $r = \frac{x-x_n}{h}$; h is step length.

1. The population of a town is given by the following data

Year	1971	1981	1991	2001	2011
Population (in thousand)	19.96	39.65	58.81	77.18	94.58

Using appropriate interpolation formula calculate the increase in the population from the year 1975 to 2005.

Sol: a) To find $f(1975) \Rightarrow x = 1975$

Here $h = 10$, $x_0 = 1971$

$$r = \frac{x - x_0}{h}$$

$$r = \frac{1975 - 1971}{10}$$

$$r = 0.4$$

x	y	$I D$	$II D$	$III D$	$IV D$
$x_0 = 1971$	$y_0 = 19.96$				
1981	39.65	$\Delta y_0 = 19.69$	$\Delta^2 y_0 = -0.53$	$\Delta^3 y_0 = -0.26$	
1991	58.81	19.16	-0.79		$\Delta^4 y_0 = 0.08$ $= \nabla^4 y_n$
2001	77.18	18.37	$\nabla^2 y_n = -0.97$	$\nabla^3 y_n = -0.18$	
$x_n = 2011$	$y_n = 94.58$	$\nabla y_n = 17.40$			

By **NFIF**,

$$y = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 y_0$$

$$y = 19.96 + (0.4)(19.69) + \frac{(0.4)(0.4-1)}{2} (-0.53) + \frac{(0.4)(0.4-1)(0.4-2)}{6} (-0.26) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24} (0.08)$$

$$y = 27.8797$$

Thus, **$f(1975) = 27.8797$**

b) To find $f(2005) \Rightarrow x = 2005$

Here $h = 0.05$, $x_n = 2011$

$$r = \frac{x-x_n}{h}$$

$$r = \frac{2005-2011}{10}$$

$$r = -0.6$$

By NBIF,

$$y = y_n + r\nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 y_n$$

$$y = 94.58 + (-0.6)(17.40) + \frac{(-0.6)(-0.6+1)}{2} (-0.97) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6} (-0.18) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24} (0.08)$$

$$y = 84.2638$$

Thus, $f(2015) = 84.2638$

$$\begin{aligned} \text{The increase in population from the year 1975 to 2005} &= f(1975) - f(2005) \\ &= 84.2638 - 27.8797 \\ &= 56.3841 \text{ (in thousands)} \end{aligned}$$

Interpolation formula for unequal intervals

Newton's Divided Difference Formula or Newton's general interpolation formula

If $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ be a set of values of an unknown function $f(x)$ corresponding to the values $x_0, x_1, x_2, \dots, x_n$ at unequal intervals, then

$$y = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \dots \\ + (x - x_0)(x - x_1) \dots (x - x_{n-1})f(x_0, x_1, \dots, x_n)$$

Here $f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$; $f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{(x_2 - x_0)}$;

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{(x_3 - x_0)}$$

Problems

1. Use NDDF to find $f(43)$, given that

x	40	42	44	45
$f(x)$	43833	46568	49431	50912

Sol:

x	$f(x)$	<i>I DD</i>	<i>II DD</i>	<i>III DD</i>
$x_0 = 40$	$f(x_0) = 43833$	$f(x_0, x_1) = \frac{46568 - 43833}{42 - 40} = 1367.5$		
$x_1 = 42$	$f(x_1) = 46568$		$f(x_0, x_1, x_2) = \frac{1431.5 - 1367.5}{44 - 40} = 16$	
$x_2 = 44$	$f(x_2) = 49431$	$f(x_1, x_2) = \frac{49431 - 46568}{44 - 42} = 1431.5$		$f(x_0, x_1, x_2, x_3) = \frac{16.5 - 16}{45 - 40} = 0.1$
$x_3 = 45$	$f(x_3) = 50912$	$f(x_2, x_3) = \frac{50912 - 49431}{45 - 44} = 1481$	$f(x_1, x_2, x_3) = \frac{1481 - 1431.5}{45 - 42} = 16.5$	



By NDDF,

$$y = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

$$y = 43833 + (43 - 40)(1367.5) + (43 - 40)(43 - 42)(16) + (43 - 40)(43 - 42)(43 - 44)(0.1)$$

$$y = 47983.2$$

Thus, $f(43) = 47983.2$

Lagrange's formula for interpolation

If $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ be a set of values of an unknown function $f(x)$ corresponding to the values $x_0, x_1, x_2, \dots, x_n$ not necessarily at equal intervals, then

$$y = f(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} y_1 + \dots + \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_2) \dots (x_n - x_{n-1})} y_n$$

Lagrange's inverse interpolation formula for $x = f(y)$ is

$$x = f(y) = \frac{(y - y_1)(y - y_2) \dots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \dots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \dots (y - y_n)}{(y_1 - y_0)(y_1 - y_2) \dots (y_1 - y_n)} x_1 + \dots + \frac{(y - y_0)(y - y_1) \dots (y - y_{n-1})}{(y_n - y_0)(y_n - y_2) \dots (y_n - y_{n-1})} x_n$$

Problems

1. Apply Lagrange's interpolation formula to find $y(11)$ from the following data

x	2	5	8	14
y	94.8	87.9	81.3	68.7

Sol: Given $x_0 = 2$ $x_1 = 5$ $x_2 = 8$ $x_3 = 14$
 $y_0 = 94.8$ $y_1 = 87.9$ $y_2 = 81.3$ $y_3 = 68.7$

To find $y(11) \Rightarrow x = 11$

Wkt

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$
$$y = \frac{(11 - 5)(11 - 8)(11 - 14)}{(2 - 5)(2 - 8)(2 - 14)} (94.8) + \frac{(11 - 2)(11 - 8)(11 - 14)}{(5 - 2)(5 - 8)(5 - 14)} (87.9) + \frac{(11 - 2)(11 - 5)(11 - 14)}{(8 - 2)(8 - 5)(8 - 14)} (81.3) + \frac{(11 - 2)(11 - 5)(11 - 8)}{(14 - 2)(14 - 5)(14 - 8)} (68.7)$$

$$y = 74.925$$

$$y(11) = 74.925$$

2. Fit an interpolating polynomial for the following data

x	0	1	2	5
y	2	3	12	147

Sol: Given $x_0 = 0$ $x_1 = 1$ $x_2 = 2$ $x_3 = 5$
 $y_0 = 2$ $y_1 = 3$ $y_2 = 12$ $y_3 = 147$

Wkt

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$y = \frac{(x - 1)(x - 2)(x - 5)}{(0 - 1)(0 - 2)(0 - 5)} (2) + \frac{(x - 0)(x - 2)(x - 5)}{(1 - 0)(1 - 2)(1 - 5)} (3) + \frac{(x - 0)(x - 1)(x - 5)}{(2 - 0)(2 - 1)(2 - 5)} (12)$$

$$+ \frac{(x - 0)(x - 1)(x - 2)}{(5 - 0)(5 - 1)(5 - 2)} (147)$$

$$\text{Wkt } (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc$$

$$y = \frac{(x^3 - 8x^2 + 17x - 10)}{(-10)}(2) + \frac{(x^3 - 7x^2 + 10x)}{(4)}(3) + \frac{(x^3 - 6x^2 + 5x)}{(-6)}(12) + \frac{(x^3 - 3x^2 + 2x)}{(60)}(147)$$

$$y = \frac{-(x^3 - 8x^2 + 17x - 10)}{5} + \frac{(3x^3 - 21x^2 + 30x)}{4} - \frac{(2x^3 - 12x^2 + 10x)}{1} + \frac{(147x^3 - 441x^2 + 294x)}{60}$$

$$y = \frac{-12(x^3 - 8x^2 + 17x - 10) + 15(3x^3 - 21x^2 + 30x) - 60(2x^3 - 12x^2 + 10x) + (147x^3 - 441x^2 + 294x)}{60}$$

$$y = \frac{-12x^3 + 96x^2 - 204x + 120 + 45x^3 - 315x^2 + 450x - 120x^3 + 720x^2 - 600x + 147x^3 - 441x^2 + 294x}{60}$$

$$y = \frac{60x^3 + 60x^2 - 60x + 120}{60}$$

$y = x^3 + x^2 - x + 2$, is the required polynomial.

Numerical Integration

The process of obtaining approximate value of the definite integration $I = \int_a^b y dx$ without actually integrating function but only using the value of 'y' at some point of x is equally placed over $[a, b]$.

Simpson's $\frac{1}{3}$ rule

Formula:

$$\int_a^b y dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2})]$$

Where, $h = \frac{b-a}{n}$

Note: To apply $\frac{1}{3}$ rule n must be multiple of 2.

Problems

1. Evaluate $\int_0^6 3x^2 dx$ dividing the interval $[0,6]$ in 6 equal parts (7 ordinate) by applying Simpson's $\frac{1^{rd}}{3}$ rule.

Sol: Given $a = 0$, $b = 6$, $y = 3x^2$

$$\text{Now, } h = \frac{b-a}{n}$$

$$h = \frac{6-0}{6}$$

$$h = 1, n = 6$$

x	0	1	2	3	4	5	6
$y = 3x^2$	0	3	12	27	48	75	108

Wkt,
 b

$$\int_a^b y \, dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_0^6 3x^2 \, dx = \frac{1}{3} [(0 + 108) + 4(3 + 27 + 75) + 2(12 + 48)]$$

$$\int_0^6 3x^2 \, dx = 216.$$

Simpson's $\frac{3^{th}}{8}$ rule

Formula:

$$\int_a^b y \, dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})]$$

$$\text{Where, } h = \frac{b-a}{n}$$

Note: To apply $\frac{3^{th}}{8}$ rule n must be multiple of 3.

Problems

1. Evaluate $\int_0^1 \frac{1}{(1+x)} dx$ by taking 7 ordinate using Simpson's $\frac{3^{th}}{8}$ rule. Hence deduce the value of $\log 2$.

Sol: Given $a = 0$, $b = 1$, $y = \frac{1}{(1+x)}$

Now, $h = \frac{b-a}{n}$, $n = 6$

$$h = \frac{1-0}{6}$$

$$h = \frac{1}{6}$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{(1+x)}$	1	0.8571	0.75	0.6667	0.6	0.5455	0.5

Wkt,

$$\int_a^b y dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_0^1 \frac{1}{(1+x)} dx = \frac{3 * \frac{1}{6}}{8} [(1 + 0.5) + 3(0.8571 + 0.75 + 0.6 + 0.5455) + 2(0.6667)]$$

$$\int_0^1 \frac{1}{(1+x)} dx = 0.6932$$

By integration,

$$\int_0^1 \frac{1}{1+x} dx = [\log(1+x)]_0^1$$

$$0.6932 = [\log 2 - \log 1]$$

$$\log 2 = 0.6932$$

Module-5: Numerical Methods-2

Numerical solution of ordinary differential equations of first order and first degree

- Taylor's series method.
- Modified Euler's method.
- Runge-Kutta method of fourth-order.
- Milne's predictor-corrector formula.

Self-study Adam-Bashforth method.



Numerical solution of ordinary differential equations of first order and first degree

A *numerical method* can be used to get an accurate approximate solution to a differential equation. There are many programs and packages available for solving these differential equations. With today's computer, an accurate solution can be obtained rapidly. In this chapter we focus on basic numerical methods for solving initial value problems.

Analytical methods, when available, generally enable to find the value of y for all values of x . Numerical methods, on the other hand, lead to the values of y corresponding only to some finite set of values of x . That is the solution is obtained as a table of values, rather than as continuous function. Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution inevitably involves an error which should be small but may, if it is not controlled, swamp the true solution. Therefore we must be concerned with two aspects of numerical solutions of ODEs: both the method itself and its accuracy.

The most general form of an ODE of first order and first degree is given by

$$\frac{dy}{dx} = f(x, y) \text{ where } y(x_0) = y_0$$

Let x be an independent variable and y be dependent variable .

Let us consider the differential equation $\frac{dy}{dx} = f(x, y) \text{ where } y(x_0) = y_0$ --(1)

If particular values are given to the constants then the resulting solution is called a particular solution.

To obtain a particular solution from the general solution (1), we must be given initial conditions so that the constants can be determined. If all the initial conditions are specified at the same value of x then the problem is termed as initial value problem. If the conditions are specified at more than one value of x , then the problem is termed as boundary value problem.

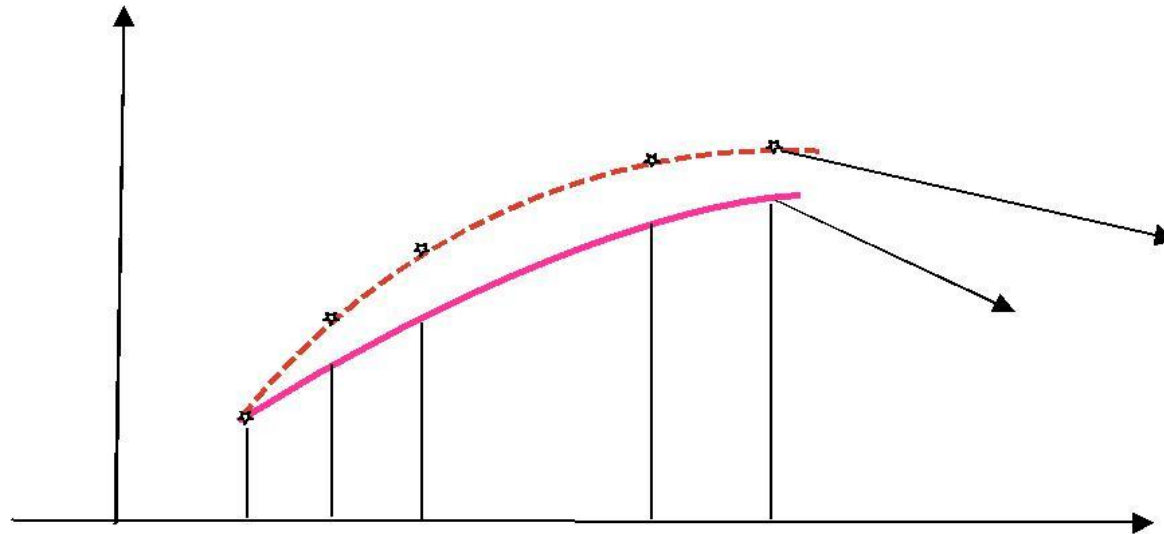
Though there are many analytical methods for finding the solution of the equation of the form (1), there exist large number of ODE's whose solution cannot be obtained by the known analytical methods. In such cases, we use numerical methods to get an approximate solution of a given differential equation under the prescribed conditions

Consider the first order differential equation $\frac{dy}{dx} = f(x, y)$

Let $y(x_0), y(x_1), y(x_2), y(x_3), \dots, y(x_m)$ be the solution values at the points $x_0, x_1, x_2, x_3, \dots, x_m$

We wish to find the approximate values y_0, y_1, \dots, y_m to these solution values.

Let the initial condition be $y(x_0) = y_0$. Let the exact solution $y(x)$ of the given differential equation be represented by a continuous curve. Divide the interval (x_0, x_m) on which the solution is derived into a finite number of equispaced subintervals.



Approximate solution
Exact solution

For each x_i , the approximate values of the dependent variable $y(x)$ are calculated using a suitable recursive formula. These values are y_0, y_1, \dots, y_m and these are shown by points. Computation of these approximate values is known as Numerical solution of the Differential equation.

An **Initial Value Problem** (IVP) consists of a differential equation and a condition which the solution must satisfy (or several conditions referring to the same value of x if the differential equation is of higher order).

$$\frac{dy}{dx} = f(x, y) \text{ where } y(x_0) = y_0 \text{ --- (1)}$$

The following methods are used to solve the IVP (1).

1. **Taylor's Series Method**
2. **Modified Euler's Method**
3. **Runge - Kutta Method**
4. **Milne's Method**

Taylor's Series Method

Let $y=f(x)$ be a solution of the equation Expanding it by Taylor's series about $x - x_0$ we get

$$f(x) = y_0 + \frac{(x - x_0)}{1!} y_0^1 + \frac{(x - x_0)^2}{2!} y_0^2 + \frac{(x - x_0)^3}{3!} y_0^3 + \dots$$

1. Using the Taylor's series method, find an approximate solution correct to four decimals at $x=0.1$ for the IVP $\frac{dy}{dx} = 2y + 3e^x, y(0) = 0$

Sol:

$$y_1 = 2y + 3e^x$$

$$y_1(0) = 0 + 3e^0 = 3$$

Differentiating w.r.t x we get

$$y_2 = 2y_1 + 3e^x$$

$$y_2(0) = 2 * 3 + 3e^0 = 9$$

$$y_3 = 2y_2 + 3e^x$$

$$y_4 = 2y_3 + 3e^x$$

Wkt,

$$f(x) = y_0 + \frac{(x-x_0)}{1!} y_0^1 + \frac{(x-x_0)^2}{2!} y_0^2 + \frac{(x-x_0)^3}{3!} y_0^3 + \dots$$

Where $x=0.1$ and $x_0 = 0$ we get

$$y_1 = f(0.1) = 0 + \frac{0.1}{1!} (3) + \frac{0.1^2}{2!} (9) + \frac{0.1^3}{3!} (21) + \frac{0.1^4}{4!} (45) + \dots$$

$$\mathbf{y(0.1)=0.3487}$$

$$y_3(0) = 2 * 9 + 3e^0 = 21$$

$$y_4(0) = 2 * 21 + 3e^0 = 45$$

Modified Euler's Method

Consider the IVP $\frac{dy}{dx} = f(x, y), y(0) = y_0$ --- (1)

To determine the solution of this problem at $x_n = x_0 + nh$ bu using Euler's method.

$$y_n^{(E)} = y_{n-1} + hf(x_{n-1}, y_{n-1}) \text{ ---(2)}$$

The expression (2) gives an approximate value of y at x_n . . To improve the approximation the following formula has been suggested

$$y_n = y_{n-1} + \frac{h}{2} \left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(E)}) \right] \text{ ---(3)}$$

The method of computing y_n using (3) is known as Modified Euler's method.

The process of improving the approximation can be continued by obtaining replacing $y_n^{(1)}, y_n^{(2)}, y_n^{(3)}$ until the desired degree of accuracy is obtained.

First approximation $y_1^{(E)} = y_0 + hf(x_0, y_0)$

Second approximation $y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(E)})]$

Third approximation $y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$ and so on.

1. Using the modified Euler's method, solve the IVP $\frac{dy}{dx} = \frac{1}{x+y}, y(0) = 1$ at points $x=0.5$ and $x=1$ in steps of length $h=0.5$. Carry out two modifications at each step.

Sol:

$$\frac{dy}{dx} = f(x, y) = \frac{1}{x+y} \quad x_0 = 0, y_0 = 1 \text{ taking } h = 0.5$$

$$y_1^{(E)} = y_0 + hf(x_0, y_0) = y_0 + h \frac{1}{x_0 + y_0} = 1 + \frac{0.5}{0+1} = \mathbf{1.5}$$

$$\begin{aligned} \text{First modification } y_1^{(1)} &= y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(E)}) \right] \\ &= y_0 + \frac{h}{2} \left[\frac{1}{x_0 + y_0} + \frac{1}{x_0 + y_1^{(E)}} \right] \\ &= 1 + \frac{0.5}{2} \left[\frac{1}{0+1} + \frac{1}{0.5+1.5} \right] = \mathbf{1.375} \end{aligned}$$

$$\begin{aligned} \text{Second modification } y_1^{(2)} &= y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] \\ &= y_0 + \frac{h}{2} \left[\frac{1}{x_0 + y_0} + \frac{1}{x_0 + y_1^{(1)}} \right] \\ &= 1 + \frac{0.5}{2} \left[\frac{1}{0+1} + \frac{1}{0.5+1.375} \right] = \mathbf{1.3833} \end{aligned}$$

Next, to compute the solution $y_2 = y(1)$ and let us consider $x_0 = 0.5$, $y_0 = 1.3833$

$$y_1^{(E)} = y_0 + hf(x_0, y_0) = y_0 + h \frac{1}{x_0 + y_0} = 1.3833 + \frac{0.5}{0.5 + 1.3833} = \mathbf{1.6488}$$

$$\begin{aligned} \text{First modification } y_1^{(1)} &= y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(E)}) \right] \\ &= y_0 + \frac{h}{2} \left[\frac{1}{x_0 + y_0} + \frac{1}{x_0 + y_1^E} \right] \\ &= 1.3833 + \frac{0.5}{2} \left[\frac{1}{0.5 + 1.3833} + \frac{1}{1 + 1.6488} \right] = \mathbf{1.6104} \end{aligned}$$

$$\begin{aligned} \text{Second modification } y_1^{(2)} &= y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] \\ &= y_0 + \frac{h}{2} \left[\frac{1}{x_0 + y_0} + \frac{1}{x_0 + y_1^1} \right] \\ &= 1.3833 + \frac{0.5}{2} \left[\frac{1}{0.5 + 1.3833} + \frac{1}{1 + 1.6104} \right] = \mathbf{1.6118} \end{aligned}$$

The required solution are $y(0.5)=1.3833$ and $y(1)=1.6118$

RUNGE- KUTTA METHOD (R-K METHOD)

Consider the IVP $\frac{dy}{dx} = f(x, y), y(0) = y_0$ ---- (1)

To determine the solution of this problem at $x_n = x_0 + nh$ by using this method, where h is step length

According to the Euler's method , the solution at x_1 is $y_0 + hf(x_0, y_0)$.

This can be rewritten as $y_1 = y_0 + K$

Where

$$\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\ k_4 &= hf(x_0 + h, y_0 + k_3) \end{aligned}$$

$y_1 = y_0 + \frac{k_1+2k_2+2k_3+k_4}{6}$ is an approximate solution for the equation (1) at x_1 , known as Runge-Kutta method of order four.

1. Using RK method of fourth order, to find $y(0.2)$, given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ and $y(0) = 1$
take $h = 0.2$

Sol:

Given $\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0, y_0 = 1$ and $h = 0.2$ then $x_1 = x_0 + h = 0.2$

$$\begin{aligned}k_1 &= hf(x_0, y_0) \\ &= h \left(\frac{y_0 - x_0}{y_0 + x_0} \right) \\ &= (0.2) \left(\frac{1-0}{1+0} \right)\end{aligned}$$

$$k_1 = 0.2$$

$$k_2 = hf \left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} \right)$$

$$= 0.2 \left(\frac{1.1-0.1}{1.1+0.1} \right)$$

$$k_2 = \mathbf{0.1667}$$

$$k_3 = hf \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} \right)$$

$$= 0.2 \left(\frac{1.0834-0.1}{1.0834+0.1} \right)$$

$$k_3 = \mathbf{0.1662}$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.2 \left(\frac{1.1662-0.2}{1.1662+0.2} \right)$$

$$k_4 = \mathbf{0.1414}$$

Wkt,

$$y_1 = y_0 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$
$$= 1 + \frac{(0.2 + 2(0.1667 + 0.1662) + 0.1414)}{6}$$

$$y(0.2) = 1.1679$$

Milne's Method:

The method in which the construction of y_n involves the use of not only y_{n-1} but also predecessors are called multi step methods. In multi-step methods two formulas are used in conjunction with each other- one for predicting the value of y_n and the other for correcting the predicted value of y_n .

Consider the IVP $\frac{dy}{dx} = f(x, y) \text{ --- (1)}$

Let $y_0=y(x_0), y_1=y(x_1), y_2=y(x_2)$ and $y_3=y(x_3)$ be these known solutions.

Suppose we wish to determine the solution of equation (1) at the point $x_4 = x_3+h$.

Let us denote the required solution by $y_4=y(x_4)$.

First we predict the value of $y_4=y(x_4)$ by using Milne's predictor formula:

$$y_4^{(p)} = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \text{ --- (2)}$$

which can be computed with the help of the specified x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 .

Next we correct the value of y_4 by using the Milne's corrector formula:

$$y_4^{(c)} = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4^{(p)}) \text{ --- (3)}$$

where $f_4^{(p)} = f(x_4, y_4^{(p)})$

If we wish to have more accurate approximation for y , we employ the process repeatedly

1. Use Milne's predictor –corrector method to find the value of y at $x=0.8$, given $\frac{dy}{dx} = x - y^2$

Where $y(0)=0$, $y(0.2)=0.02$, $y(0.4)=0.0795$, $y(0.6)=0.1762$. Apply corrector formula twice.

Sol:

Here $f(x, y) = x - y^2, h = 0.2$

x	Y	$f(x, y) = x - y^2$
$x_0=0$	$y_0=0$	$f_0=0-0^2=0$
$x_1=0.2$	$y_1=0.02$	$f_1=0.2-0.02^2=0.1996$
$x_2=0.4$	$y_2=0.0795$	$f_2=0.4-0.0795^2=0.3937$
$x_3=0.6$	$y_3=0.1762$	$f_3=0.6-0.1762^2=0.5689$

Now , Milne's Predictor formula yields the predicted value of y_4 as

$$\begin{aligned}y_4^{(p)} &= y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \\ &= 0 + \frac{4 \cdot 0.2}{3} (2(0.1996) - 0.3937 + 2(0.5689))\end{aligned}$$

$$y_4^{(p)} = \mathbf{0.3048}$$

$$f_4^{(p)} = f(x_4, y_4^{(p)}) = x_4 - y_4^{(p)^2} = 0.8 - (0.30488)^2$$

$$f_4^{(p)} = \mathbf{0.7070}$$

Now , the Milne's corrector formula gives a corrected value of y_4 as

$$\begin{aligned}y_4^{(c)} &= y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4^{(p)}) \\ &= 0.0795 + \frac{0.2}{3} (0.3937 + 4(0.5689) + 0.7070)\end{aligned}$$

$$y_4^{(c)} = 0.3046$$

To apply corrector formula second time we must use corrector as predictor and substitute in $f_4^{(p)}$

$$f_4^{(p)} = f(x_4, y_4^{(p)}) = x_4 - y_4^{(p)^2} = 0.8 - (0.3046)^2$$

$$f_4^{(p)} = 0.7072$$

$$y_4^{(c)} = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4^{(p)}) = 0.0795 + \frac{0.2}{3} (0.3937 + 4(0.5689) + 0.7072)$$

$$y_4^{(c)} = 0.3046$$

Thus, $y(0.8) = 0.3046$



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Thank You