ATME College of Engineering

MATHEMATICS-II FOR MECHANICAL ENGINEERING STREAMS

(BMATC201)

ATME COLLEGE OF ENGINEERING

DEPARTMENT OF MATHEMATICS

Course objectives:

- The goal of the course **Mathematics-II for mechanical engineering streams(BMATM201)** is to
- · **Familiarize** the importance of Integral calculus and Vector calculus essential for Mechanical engineering.
- · **Analyze** Mechanical engineering problems applying Partial Differential Equations.
- · **Develop** the knowledge of solving Mechanical engineering problems numerically

Course outcome (Course Skill Set)

After successfully completing the course, the student will have a good understanding of the following topics and their applications:

- **CO1:** Apply the knowledge of multiple integrals to compute area and volume.
- **CO2:** Understand the applications of vector calculus refer to solenoidal, ir-rotational vectors, line integral and surface integral.
- **CO3:** Demonstrate partial differential equations and their solutions for physical interpretations.
- **CO4:** Apply the knowledge numerical methods in solving physical and engineering phenomena.
- **CO5:** Get familiarize with modern mathematical tools namely Mathematica/MatLab/Python/Scilab

CONTENTS

Module-1: Integral Calculus

- Multiple Integrals: Evaluation of double and triple integrals
- Evaluation of double integrals by change of order of integration, changing into polar coordinates.
- Applications to find Area and Volume by a double integral-Problems.
- Beta and Gamma functions: Definitions, properties, the relation between Beta and Gamma functions.
- Problems on Beta and Gamma functions.

Self-study: Centre of gravity.

Multiple Integral

Double Integral

$$\iint f(x,y)dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

Evaluate the following:

1. $\int_0^1 \int_x^{\sqrt{x}} xy\, dydx$

$\boldsymbol{Sol}$: $\boldsymbol{I} = \int_0^1 x\left[\frac{y^2}{2}\right]_x^{\sqrt{x}}\, dx = \frac{1}{2}\int_0^1 x\,[x - x^2]\,dx$

$$= \frac{1}{2}\int_0^1 [x^2 - x^3]\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \mathbf{\frac{1}{24}}$$

2. $\int_0^1 \int_0^y xy\,(x+y)dydx$

$$Sol: I = \int_0^1 \int_0^y \ (x^2y + y^2x)dydx$$

$$= \int_0^1 [y\frac{x^3}{3} + y^2\frac{x^2}{2}]_0^y \ dy$$

$$= \int_0^1 [\ \frac{y^4}{3} + \frac{y^4}{2}]dy$$

$$= \int_0^1 \ \frac{5y^4}{6} \ dy$$

$$= \frac{5}{6}\,[\frac{y^5}{5}]_0^1$$

$$\boldsymbol{I = \frac{1}{6}}$$

Triple integrals:

$$\iiint f(x,y,z)dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dzdydx$$

Evaluate the following:

1. $\int_0^1\int_0^x\int_0^{xz} xyz\, dzdydx$

$$Sol: \quad Let\ I = \int_0^1\int_0^x\int_0^{xz} xyz\, dzdydx$$

$$= \int_0^1\int_0^x xz\left[\frac{y^2}{2}\right]_0^{xz} dz\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x xz\ (x^2z^2 - 0)dz\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x x^3z^3\ \ dz\,dx$$

$$I = \frac{1}{2}\int_0^1 x^3 \left[\frac{z^4}{4}\right]_0^x \; dx$$

$$= \frac{1}{8}\int_0^1 x^3 \left(x^4 - 0\right) \; dx$$

$$= \frac{1}{8}\int_0^1 x^7 \; dx$$

$$= \frac{1}{8}\left[\frac{x^8}{8}\right]_0^1 = \frac{1}{64}$$

$$\therefore \boldsymbol{I} = \frac{\mathbf{1}}{\mathbf{64}}$$

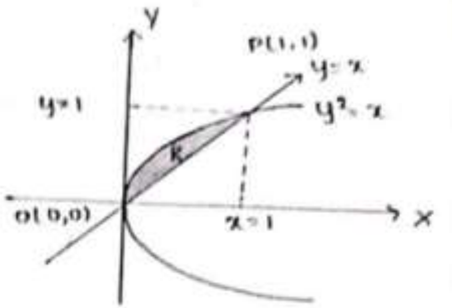
Evaluation of double integrals- change of order of integration

1.Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \; dy \, dx$ by changing the order of the integration.

Sol: $Let\ I = \int_0^1 \int_x^{\sqrt{x}} xy \; dy \, dx$

$Given\ Limits \quad x : x = 0\ to\ x = 1$

$y : \ y = x\ to\ y = \sqrt{x}\ \ or\ y^2 = x$



On changing the order of the integration.

$x : x = y^2\ to\ x = y$

$y : \ y = 0\ to\ y = 1$

$\therefore I = \int_{y=0}^{1} \int_{x=y^2}^{y} y \; (xdx)dy$

$$I = \int_{y=0}^{1} y \left[{x^2}/{2}\right]_{y^2}^{y} dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y(y^2 - y^4)dy$$

$$= \frac{1}{2}\int_{y=0}^{1} (y^3 - y^5)dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1$$

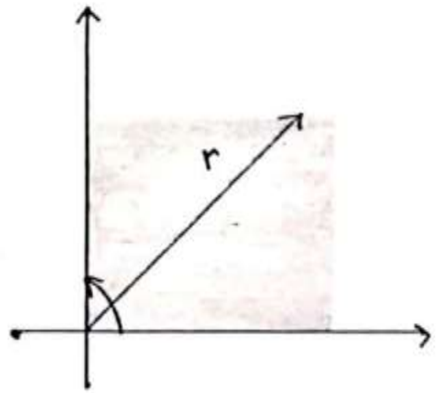
$$\therefore \boldsymbol{I = \frac{1}{24}}$$

Evaluation by changing into Polar coordinates

1. $Evaluate \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$ $by\ changing\ to\ polar\ coordinates$

Sol: $In\ polar\ we\ have\ x = rcos\theta\ , y = rsin\theta$

$\therefore\ x^2 + y^2 = r^2\ and\ dxdy = r\ dr\ d\theta$



$Since\ x\ varies\ from\ 0\ to\ \infty \Rightarrow\ r\ varies\ from\ 0\ to\ \infty$

$In\ the\ 1st\ quadrant\ \theta\ varies\ from\ 0\ to\ \frac{\pi}{2}$

$\therefore \boldsymbol{I} = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2}\ r\ dr\ d\theta$

$Put\ r^2 = t \Rightarrow 2rdr = dt\ \Rightarrow rdr = \frac{dt}{2}$

$and\ t\ also\ varies\ from\ 0\ to\ \infty$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2} \, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-1} \right]_{t=0}^{\infty} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} -(0 - 1) d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} 1 \, d\theta$$

$$= \frac{1}{2} [\, \theta \,]_{\theta=0}^{\frac{\pi}{2}}$$

$$\therefore \boldsymbol{I} = \frac{\boldsymbol{\pi}}{\boldsymbol{4}}$$

Application of Double and Triple internals

Area of Region R in the Cartesian form: $\iint_R \; dxdy$

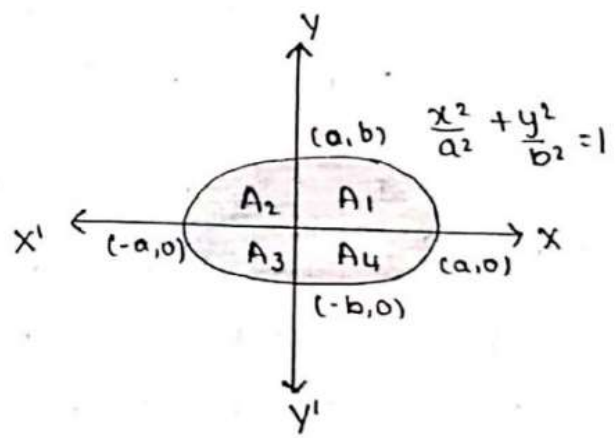
Area of Region R in the Polar form: $\iint_R \; r\,dr\,d\theta$

Volume of solid in the Cartesian form: $\iiint_V \; dx\,dy\,dz$

Volume of solid in the Polar form: $\iint_R \; 2\pi r^2 sin\theta\,dr\,d\theta$

1. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ by\ double\ integtation.$

$\boldsymbol{Sol}: Let\ \ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ is\ ellipse$



$$Area,\ \ A = 4\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\, dx$$

$$= 4\int_{x=0}^{a}[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\ dx$$

$$= \ 4\int_{x=0}^{a}\frac{b}{a}\sqrt{a^2-x^2}\ \ dx$$

$$= \frac{4b}{a}\int_{x=0}^{a}\sqrt{a^2-x^2}\ \ dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$A = \frac{4b}{a}\left[\frac{a^2}{2} sin^{-1}(1)\right]$$

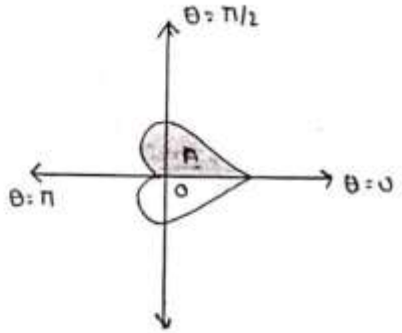
$$= \frac{4b}{a}\left[\frac{a^2}{2}\frac{\pi}{2}\right]$$

$$\therefore \boldsymbol{A = \pi ab\ sq.units}$$

2. *Find the volume generated by the revolution of the cardiode* $\mathrm{r = a(1 + cos\theta)}$ *about the initial line using double integral.*

Sol: $Wkt\ ,\ V = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+cos\theta)} 2\pi r^2 sin\theta\, dr\, d\theta$

$$V = \int_{\theta=0}^{\pi} 2\pi sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+cos\theta)} d\theta$$



$$V = \int_{\theta=0}^{\pi} 2\pi sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2\pi}{3}\int_{\theta=0}^{\pi} sin\theta\ a^3(1+\cos\theta)^3\ d\theta$$

$$= \frac{2a^3\pi}{3}\int_{\theta=0}^{\pi} sin\theta\ (1+\cos\theta)^3\ d\theta$$

$$Put\ \ 1 + cos\theta = t \Rightarrow -sin\ \theta d\theta = dt\ \ \Rightarrow sin\ \theta d\theta = -dt$$

$$if\ \theta\ varies\ from\ 0\ to\ \pi\ \ then\ \ t\ also\ varies\ from\ 2\ to\ 0$$

$$V = \frac{2a^3\pi}{3}\int_{t=2}^{0} (\mathrm{t})^3\ .-dt \quad = \frac{-2a^3\pi}{3}\left[\frac{\mathrm{t}^4}{4}\right]_2^0$$

$$V = \frac{-a^3\pi}{6}(0 - 16)$$

$$\therefore \boldsymbol{V = \frac{8a^3\pi}{3}\ cubic\ units}$$

Beta and Gamma functions:

Definition:

Let n be any positive number. The ***gamma*** function of n is denoted by $\boldsymbol{\Gamma}(n)$ and it is defined as

$$\boldsymbol{\Gamma(n) = \int_{x=0}^{\infty} e^{-x} x^{n-1} dx}$$

The ***beta*** function is denoted by $\beta(m,n)$ and it is defined as

$$\boldsymbol{\beta(m,n) = \int_{x=0}^{1} x^{m-1}(1-x)^{n-1}dx \ \ , m,n > 0}.$$

Relation between Beta and Gamma functions

$$\boldsymbol{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Note:

1. $\boldsymbol{\beta(m,n) = 2\int_{x=0}^{\frac{\pi}{2}} sin^{2m-1}\theta \, cos^{2n-1}\theta \, d\theta}$
2. $\boldsymbol{\Gamma(n) = \int_{x=0}^{\infty} e^{-t^2} t^{2n-1} dt}$

Properties of Beta and Gamma function

1. $\beta(m,n) = \beta(n,m)$
2. $\Gamma(n+1) = n\Gamma(n)$
3. $\Gamma(n+1) = n!$, for any positive integer n
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
5. $\int_{x=0}^{\frac{\pi}{2}} sin^p\theta\, cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$
6. $\Gamma\left(\frac{1}{4}\right).\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

Module-2: Vector Calculus

- Vector Differentiation: Scalar and vector fields. Gradient, directional derivative.
- curl and divergence - physical interpretation, solenoidal and irrotational vector fields. Problems.
- Vector Integration: Line integrals, Surface integrals.
- Applications to work done by a force and flux.
- Statement of Green's theorem and Stoke's theorem-Problems.

Self-study: Volume integral and Gauss divergence theorem.

VECTOR CALCULUS

Vector calculus is a field of mathematics concerned with multivariate real analysis of vectors in two or more dimensions. It consists of set of problems solving techniques very useful for engineering and physics.

SCALAR AND VECTOR POINT FUNCTIONS

Let $\vec{F}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$ be a 'Vector function'. Then for various values of t we get a set of constant vectors.

Let $\varphi = \varphi(x, y, z)$ be a 'Scalar function'. Then for various values of x, y, z we get a set of points or scalars.

Vector operation $\nabla(del)$ is defined by the equation

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

This operator has a great role in vector calculus. Laplacian operator ∇^2 is defined as follows

$$\nabla^2 = \nabla.\nabla = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$\boldsymbol{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

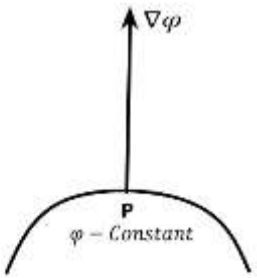
Gradient

The vector function $\nabla\varphi$ is defined as the gradient of the scalar function $\varphi = \varphi(x, y, z)$

$$i.e., \quad grad\varphi = \nabla\varphi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\varphi$$

$$\boldsymbol{grad\varphi = \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right)}$$

Geometrically, $\nabla\varphi$ represents a normal at any point P to the surface $\varphi(x, y, z) = constant$ and has a magnitude equal to the rate of change of $\varphi(x, y, z)$ along this normal. $\nabla\varphi$ is a **vector quantity**.



Note:

1. The unit normal vector $\hat{n}$ along $\nabla\varphi$ is given by $\hat{\boldsymbol{n}} = \frac{\nabla\boldsymbol{\varphi}}{|\nabla\boldsymbol{\varphi}|}$ or $\hat{n} = \frac{\nabla f}{|\nabla f|}$

2. The component of $\nabla\varphi$ in the direction of a unit vector $\vec{a}$ is $\nabla\varphi.\hat{n}$ and is called the *directional derivative* of φ in the direction of $\vec{a}$. Thus, the directional derivative is maximum in the direction $\nabla\varphi$ and the magnitude of this maximum is equal to $|\nabla\varphi|$.

$i.e.,\ \boldsymbol{D.D} = \boldsymbol{\nabla\varphi.\hat{n}}$ where $\hat{n} = \frac{\vec{a}}{|\vec{a}|}$

Problems

1. Find the unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at $(1,2,-1)$.

Sol: Let $\varphi = x^3 + y^3 + 3xyz - 3$

$\frac{\partial\varphi}{\partial x} = 3x^2 + 3yz$ $\frac{\partial\varphi}{\partial y} = 3y^2 + 3xz$ $\frac{\partial\varphi}{\partial z} = 3xy$

Now, $\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$

$$\nabla\varphi = (3x^2 + 3yz)\hat{\imath} + (3y^2 + 3xz)\hat{\jmath} + (3xy)\hat{k}$$

At $(1,2,-1)$

$\nabla\varphi = (3-6)\hat{\imath} + (12-3)\hat{\jmath} + (6)\hat{k}$

$\nabla\varphi = -3\hat{\imath} + 9\hat{\jmath} + 6\hat{k}$

$|\nabla\varphi| = \sqrt{(-3)^2+(9)^2+(6)^2} = \sqrt{9+81+36}$

$$|\boldsymbol{\nabla\varphi}| = \sqrt{\mathbf{126}}$$

The unit normal vector , $\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla\varphi}}{|\boldsymbol{\nabla\varphi}|} = \frac{-3\hat{\imath}+9\hat{\jmath}+6\hat{k}}{\sqrt{\mathbf{126}}}$

2. Find the directional derivative of the function $f = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$

Sol: Given $f = 4xz^3 - 3x^2y^2z$ Let $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$

$\frac{\partial f}{\partial x} = 4z^3.1 - 3y^2z.2x$ $\frac{\partial f}{\partial y} = 0 - 3x^2z.2y$ $\frac{\partial f}{\partial z} = 4x.3z^2 - 3x^2y^2.1$

$\frac{\partial f}{\partial x} = 4z^3 - 6xy^2z$ $\frac{\partial f}{\partial y} = -6x^2yz$ $\frac{\partial f}{\partial z} = 12xz^2 - 3x^2y^2$

Now, $\nabla f = \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right)$

$\nabla f = (4z^3 - 6xy^2z)\hat{\imath} + (-6x^2yz)\hat{\jmath} + (12xz^2 - 3x^2y^2)\hat{k}$

At $(2,-1,2)$

$\nabla f = (32 - 24)\hat{\imath} + (48)\hat{\jmath} + (96 - 12)\hat{k}$

$$\boldsymbol{\nabla f = 8\hat{\imath} + 48\hat{\jmath} + 84\hat{k}}$$

Also, $\boldsymbol{\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}}$

$$|\vec{a}| = \sqrt{2^2 + (-3)^2 + 6^2}$$

$$|\vec{a}| = \sqrt{4 + 9 + 36}$$

$$|\vec{a}| = \sqrt{49}$$

$$\boldsymbol{|\vec{a}| = 7}$$

$$\therefore \ \hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\boldsymbol{2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}}}{\boldsymbol{7}}$$

$$\boldsymbol{D.D = \nabla f.\hat{n}}$$

$$D.D = \left(8\hat{\imath} + 48\hat{\jmath} + 84\hat{k}\right).\frac{\left(2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}\right)}{7}$$

$$D.D = \frac{(8)(2) + (48)(-3) + (84)(6)}{7}$$

$$D.D = \frac{16 - 144 + 504}{7}$$

$$\boldsymbol{D.D = \frac{376}{7}}$$

3. Find the angle between the surfaces or normal surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.

Sol: Given $\varphi = x^2 + y^2 + z^2 - 9$ $\qquad$ $\psi = x^2 + y^2 - z - 3$

$$\frac{\partial\varphi}{\partial x} = 2x \qquad \frac{\partial\psi}{\partial x} = 2x$$

$$\frac{\partial\varphi}{\partial y} = 2y \qquad \frac{\partial\psi}{\partial y} = 2y$$

$$\frac{\partial\varphi}{\partial z} = 2z \qquad \frac{\partial\psi}{\partial z} = -1$$

Wkt, $\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$

$$\nabla\varphi = (2x)\hat{\imath} + (2y)\hat{\jmath} + (2z)\hat{k}$$

At $(2,-1,2)$

$$\nabla\varphi = (2.2)\hat{\imath} + (2.(-1))\hat{\jmath} + (2.2)\hat{k}$$

$$\boldsymbol{\nabla\varphi = 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}$$

$$|\nabla\varphi| = \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36}$$

$$\boldsymbol{|\nabla\varphi| = 6}$$

Also $\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\imath} + \frac{\partial\psi}{\partial y}\hat{\jmath} + \frac{\partial\psi}{\partial z}\hat{k}$

$$\nabla\psi = (2x)\hat{\imath} + (2y)\hat{\jmath} + (-1)\hat{k}$$

At $(2,-1,2)$

$$\nabla\psi = (2.2)\hat{\imath} + (2.(-1))\hat{\jmath} + (-1)\hat{k}$$

$$\boldsymbol{\nabla\psi = 4\hat{\imath} - 2\hat{\jmath} - \hat{k}}$$

$$|\nabla\psi| = \sqrt{4^2 + (-2)^2 + (-1)^2}$$

$$|\nabla\psi| = \sqrt{\mathbf{21}}$$

Angle between surface,

$$cos\theta = \widehat{n_1}.\widehat{n_2}$$

$$cos\theta = \frac{\nabla\varphi}{|\nabla\varphi|}.\frac{\nabla\psi}{|\nabla\psi|}$$

$$cos\theta = \frac{(\mathbf{4}\hat{\imath} - \mathbf{2}\hat{\jmath} + \mathbf{4}\hat{k})}{6}.\frac{(\mathbf{4}\hat{\imath} - \mathbf{2}\hat{\jmath} - \hat{k})}{\sqrt{21}}$$

$$cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$cos\theta = \frac{8}{3\sqrt{21}}$$

$$\boldsymbol{\theta = cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Divergence of a vector function

The divergence of a vector function $\vec{F} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$, where f_1, f_2, f_3 are functions of x, y, z.It is denoted by $div\vec{F}$ and is defined as

$$div\vec{F} = \nabla.\vec{F}$$

$$div\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).(f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k})$$

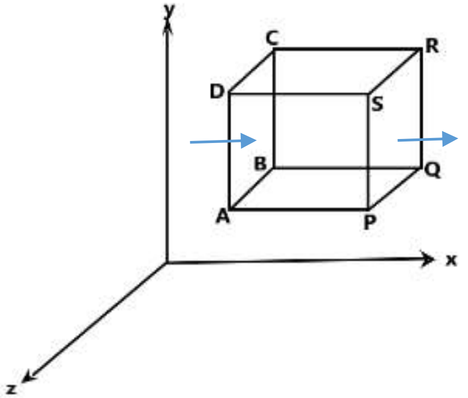
$$\boldsymbol{div\vec{F} = \nabla.\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

Clearly $\boldsymbol{div\vec{F}}$ is a scalar quantity.

Physical interpretation of Divergence

Let us consider the motion of the fluid. Consider a small rectangular parallelepiped with edges $\delta_x, \delta_y, \delta_z$ parallel to the axes in the mass of fluid.

Let $\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}$ be the velocity of the fluid at (x, y, z).



Amount of the fluid flowing in through the face ABCD per unit time

$$= Velocity\ X\ Area\ of\ the\ face$$

$$= V_x\,\delta y\delta z$$

Amount of the fluid flowing out through the face PQRS per unit time

$$= \left[V_x + \frac{\partial V_x}{\partial x}\,\delta x\right]\delta y\delta z$$

∴ The net decrease in the amount of fluid across these two faces is

$$= \left[V_x + \frac{\partial V_x}{\partial x}\,\delta x\right]\delta y\delta z - V_x\delta y\delta z$$

$$= \left[V_x + \frac{\partial V_x}{\partial x}\,\delta x - V_x\right]\delta y\delta z = \boldsymbol{\frac{\partial V_x}{\partial x}\,\delta x\delta y\delta z}$$

Similarly, the decrease in amount of fluid due to flow along the y − axis $= \frac{\partial V_y}{\partial y}\ \delta x \delta y \delta z$

The decrease in amount of fluid due to flow along the z − axis $= \frac{\partial V_z}{\partial z}\ \delta x \delta y \delta z$

Total decrease in amount of fluid inside the parallelepiped per unit time

$$= \left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right]\ \delta x \delta y \delta z$$

Hence the ratio of loss of fluid per unit volume $= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$= \left[\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right].\left[V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}\right]$$

$$= \nabla.\vec{V}$$

$$= div\vec{V}$$

Hence $div\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid. If $\boldsymbol{div\vec{V} = 0}$ everywhere in some region of space, then $\vec{V}$ is called ***Solenoidal Vector function*** and the fluids said to be *incompressible* i.e., there is no gain or loss in the volume element

Curl of a vector function

The curl of a vector function $\vec{F} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$ is denoted by $curl\vec{F}$ and is defined as $\boldsymbol{curl\vec{F} = \nabla X\vec{F}}$

$$curl\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\boldsymbol{curl\vec{F} = \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\hat{\imath} - \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right]\hat{\jmath} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\hat{k}}$$

Clearly $\boldsymbol{curl\vec{F}}$ is a vector quantity.

Physical interpretation of Curl

Consider a rigid body rotating about a fixed axis through origin. Let the uniform angular velocity be $\vec{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}$, w_1, w_2, w_3 are constants. The velocity $\vec{V}$ of any point P(x, y, z) on the body is given by $\vec{V} = \vec{w}\ X\ \vec{r}$, where $\vec{r}$ is the position vector of P.

Let $\vec{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Consider $\vec{V} = \vec{w}\ X\ \vec{r}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$\vec{V} = (w_2 z - w_3 y)\hat{\imath} - (w_1 z - w_3 x)\hat{\jmath} + (w_1 y - w_2 x)\hat{k}$$

$$curl\vec{V} = \nabla X\vec{V}$$

$$curl\vec{V} = \nabla\ X\left[(w_2z - w_3y)\hat{\imath} - (w_1z - w_3x)\hat{\jmath} + (w_1y - w_2x)\hat{k}\right]$$

$$curl\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (w_2z - w_3y) & (w_3x - w_1z) & (w_1y - w_2x) \end{vmatrix}$$

$$curl\vec{V} = [w_1 - (-w_1)\ ]\hat{\imath} - [-w_2 - w_2]\hat{\jmath} + [w_3 - (-w_3)]\hat{k}$$

$$curl\vec{V} = [2w_1\ ]\hat{\imath} + [2w_2]\hat{\jmath} + [2w_3]\hat{k}$$

$$curl\vec{V} = 2\left[w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}\right]$$

$curl\vec{V} = 2\vec{w}$

$$\vec{\boldsymbol{w}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{curl}\vec{\boldsymbol{V}}$$

Thus the angular velocity of rotation at any point is equal to half of the curl of the velocity.

Note:

1. If $\boldsymbol{div\vec{F} = 0}$, then we say that $\vec{F}$ is **Solenoidal** vector.
2. If $\boldsymbol{curl\vec{F} = 0}$, then we say that $\vec{F}$ is **irrotational** vector.
3. Irrotational vector field is called as conservative field or potential field.
4. When $\vec{F}$ is irrotational there always exist a scalar point function such that $\boldsymbol{\nabla\varphi = \vec{F}}$, then φ is called a scalar potential of vector $\vec{F}$.

Problems

1. If $\vec{F} = xyz\hat{\imath} + 3x^2y\hat{\jmath} + (xz^2 - y^2z)\hat{k}$ **,** find the $div\vec{F}$ and $curl\vec{F}$ at $(2,-1,1)$.

Sol: Given $\overrightarrow{F} = xyz\hat{\imath} + 3x^2y\hat{\jmath} + (xz^2 - y^2z)\hat{k}$

Wkt $div\vec{F} = \nabla.\vec{F}$

$$div\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$div\vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$div\vec{F} = yz.1 + 3x^2.1 + (x.2z - y^2.1)$$

$$div\vec{F} = yz + 3x^2 + 2xz - y^2$$

At $(2,-1,1)$

$$div\vec{F} = (-1)(1) + 3.(2)^2 + 2(2)(1) - (-1)^2$$

$$div\vec{F} = -1 + 12 + 4 - 1$$

$\boldsymbol{div\vec{F}}$ **=14**

Also, $curl\vec{F} = \nabla \, X \, \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xyz) & (3x^2y) & (xz^2 - y^2z) \end{vmatrix}$$

$$curl\vec{F} = \left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]\hat{k}$$

$$curl\vec{F} = [(-2yz) - 0]\hat{\imath} - [(z^2 - 0) - (xy)]\hat{\jmath} + [(6xy) - (xz)]\hat{k}$$

$$curl\vec{F} = [-2yz]\hat{\imath} - [z^2 - xy]\hat{\jmath} + [6xy - xz]\hat{k}$$

At (2,-1,1)

$$curl\vec{F} = [-2(-1)(1)]\hat{\imath} - [1^2 - 2(-1)]\hat{\jmath} + [6(2)(-1) - (2)(1)]\hat{k}$$

$$\boldsymbol{curl\vec{F} = 2\hat{\imath} - 3\hat{\jmath} - 14\hat{k}}$$

2. Find $div\vec{F}$ and $curl\vec{F}$ where $\vec{F} = grad(x^3 + y^3 + z^3 - 3xyz)$.

Sol: Let $\varphi = x^3 + y^3 + z^3 - 3xyz$

$\vec{F} = grad\varphi$

$\vec{F} = \nabla\varphi$

$$\vec{F} = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$\boldsymbol{\vec{F} = (3x^2 - 3yz)\hat{\imath} + (3y^2 - 3xz)\hat{\jmath} + (3z^2 - 3xy)\hat{k}}$$

Now, $div\vec{F} = \nabla.\vec{F}$

$$div\vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$div\vec{F} = (6x - 0) + (6y - 0) + (6z - 0)$$

$$\boldsymbol{div\vec{F} = 6x + 6y + 6z}$$

$$curl\vec{F} = \nabla\ X\ \vec{F}$$

$$curl\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$$

$$curl\vec{F} = \left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right]\hat{k}$$

$$curl\vec{F} = [(0 - 3x) - (0 - 3x)]\hat{\imath} - [(0 - 3y) - (0 - 3y)]\hat{\jmath} + [(0 - 3z) - (0 - 3z)]\hat{k}$$

$$curl\vec{F} = [-3x + 3x]\hat{\imath} - [-3y + 3y]\hat{\jmath} + [-3z + 3z]\hat{k}$$

$$curl\vec{F} = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k}$$

$$\boldsymbol{curl\vec{F} = \vec{0}}$$

3. Show that $\vec{F} = \frac{x\hat{\imath}+y\hat{\jmath}}{(x^2+y^2)}$ is both solenoidal and irrotational.

Sol: Given $\vec{F} = \frac{x}{(x^2+y^2)}\hat{\imath} + \frac{y}{(x^2+y^2)}\hat{\jmath}$

Wkt, $div\vec{F} = \nabla.\vec{F}$

$$div\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)}\right)+\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2)}\right)$$

$$div\vec{F} = \frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2} + \frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}$$

$$div\vec{F} = \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\boldsymbol{div\vec{F} = 0}$$

$\therefore$ $\vec{F}$ **is Solenoidal.**

Now, $curl\vec{F} = \nabla \, X \, \vec{F}$

$$curl\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2)} & \frac{y}{(x^2+y^2)} & 0 \end{vmatrix}$$

$$curl\vec{F} = \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{y}{(x^2+y^2)}\right)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2)}\right)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)}\right) - \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)}\right)\right]\hat{k}$$

$$curl\vec{F} = [(0)-(0)]\hat{\imath} - [(0)-(0)]\hat{\jmath} + \left[\frac{-2xy}{(x^2+y^2)} + \frac{2xy}{(x^2+y^2)}\right]\hat{k}$$

$$curl\vec{F} = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k}$$

$$\boldsymbol{curl\vec{F} = \vec{0}}$$

$\therefore \vec{F}$ is irrotational.

4. P.T $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ is irrotational. Also find a scalar point function φ such that $\vec{F} = \nabla\varphi$.

Sol: Given $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

$curl\vec{F} = \nabla \, X \, \vec{F}$

$$curl\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$curl\vec{F} = \left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right]\hat{i} - \left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right]\hat{j} + \left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]\hat{k}$$

$$curl\vec{F} = [(1)-(1)]\hat{i} - [(1)-(1)]\hat{j} + [(1)-(1)]\hat{k}$$

$$curl\vec{F} = 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$\boldsymbol{curl\vec{F} = \vec{0}}$

$\therefore$ $\vec{F}$ is irrotational.

<u>To find φ</u>

Consider $\nabla\varphi = \vec{F}$

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = (y+z)\hat{\imath} + (z+x)\hat{\jmath} + (x+y)\hat{k}$$

$$\frac{\partial\varphi}{\partial x} = y+z \qquad \frac{\partial\varphi}{\partial y} = z+x \qquad \frac{\partial\varphi}{\partial z} = x+y$$

Integrating we get

$$\varphi = (y+z)\int 1\,dx \qquad \varphi = (z+x)\int 1\,dy \qquad \varphi = (x+y)\int 1\,dz$$

$$\varphi = (y+z)x + f(y,z) \qquad \varphi = (z+x)y + f(x,z) \qquad \varphi = (x+y)z + f(x,y)$$

$$\varphi = xy + xz + f(y,z) \qquad \varphi = yz + xy + f(x,z) \qquad \varphi = xz + yz + f(x,y)$$

$\boldsymbol{curl\vec{F} = \vec{0}}$

$\therefore$ **$\vec{F}$ is irrotational.**

<u>To find φ</u>

Consider $\nabla\varphi = \vec{F}$

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = (y+z)\hat{\imath} + (z+x)\hat{\jmath} + (x+y)\hat{k}$$

$$\frac{\partial\varphi}{\partial x} = y + z \qquad \frac{\partial\varphi}{\partial y} = z + x \qquad \frac{\partial\varphi}{\partial z} = x + y$$

Integrating we get

$$\varphi = (y+z)\int 1\,dx \qquad \varphi = (z+x)\int 1\,dy \qquad \varphi = (x+y)\int 1\,dz$$

$$\varphi = (y+z)x + f(y,z) \qquad \varphi = (z+x)y + f(x,z) \qquad \varphi = (x+y)z + f(x,y)$$

$$\varphi = xy + xz + f(y,z) \qquad \varphi = yz + xy + f(x,z) \qquad \varphi = xz + yz + f(x,y)$$

$\boldsymbol{curl\vec{F} = \vec{0}}$

$\therefore$ $\vec{F}$ **is irrotational.**

<u>To find φ</u>

Consider $\nabla\varphi = \vec{F}$

$$\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = (y+z)\hat{\imath} + (z+x)\hat{\jmath} + (x+y)\hat{k}$$

$$\frac{\partial\varphi}{\partial x} = y + z \qquad \frac{\partial\varphi}{\partial y} = z + x \qquad \frac{\partial\varphi}{\partial z} = x + y$$

Integrating we get

$$\varphi = (y+z)\int 1\,dx \qquad \varphi = (z+x)\int 1\,dy \qquad \varphi = (x+y)\int 1\,dz$$

$$\varphi = (y+z)x + f(y,z) \qquad \varphi = (z+x)y + f(x,z) \qquad \varphi = (x+y)z + f(x,y)$$

$$\varphi = xy + xz + f(y,z) \qquad \varphi = yz + xy + f(x,z) \qquad \varphi = xz + yz + f(x,y)$$

$\therefore$ $\boldsymbol{\varphi = xy + xz + yz}$ **,** where $f(y,z) = yz$, $f(x,z) = xz$, $f(x,y) = xy$

VECTOR INTEGRATION

Line Integral

Consider a curve C in space which consists of infinitesimally small elements of length dr. Then the line integral of a vector $\vec{A}(x,y,z)$ along the curve C is defined to be the sum of the scalar products of $\vec{A}$,$d\vec{r}$ and is represented by $\int_C \vec{A}.d\vec{r}$.

If $\vec{F}$ is the force acted upon by a particle in displacing it along the curve C then $\int_C \vec{F}.d\vec{r}$ represents the total work done by a force , it also represents the circulation of $\vec{F}$ about C where $\vec{F}$ represents the velocity of a fluid.

$\vec{F}$ is said to be irrotational if $\oint_C \vec{F}.d\vec{r} = 0$.

Problems

1. If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, evaluate $\int_C \vec{F}\,.\,d\vec{r}$ where C is the curve $y = x^3$ from the point (1,1) to the point (2,8).

Sol: Given $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Consider, $\vec{F}.d\vec{r} = [(5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}].\left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right]$

$$\vec{F}.d\vec{r} = (5xy - 6x^2)dx + (2y - 4x)dy \quad \textbf{--- (1)}$$

In $C : y = x^3$ Points: (1,1) (2,8)

$$dy = 3x^2dx$$

(1)⇒ $\vec{F}.d\vec{r} = (5x^4 - 6x^2)dx + (2x^3 - 4x).3x^2dx$

$$\vec{F}.d\vec{r} = (5x^4 - 6x^2 + 6x^5 - 12x^3)dx$$

$$\int_C \vec{F}\,.\,d\vec{r} = \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx$$

$$\int_C \vec{F}\,.\,d\vec{r} = 5\left[\frac{x^5}{5}\right]_{x=1}^{x=2} - 6\left[\frac{x^3}{3}\right]_{x=1}^{x=2} + 6\left[\frac{x^6}{6}\right]_{x=1}^{x=2} - 12\left[\frac{x^4}{4}\right]_{x=1}^{x=2}$$

$$= (2^5 - 1) - 2(2^3 - 1) + (2^6 - 1) - 3(2^4 - 1)$$

$$= 31 - 14 + 63 - 45$$

$$\int_C \boldsymbol{\vec{F}\,.\,d\vec{r} = 35}$$

2. Evaluate $\int_C \vec{F}\,.\,d\vec{r}$ along the circle $x^2 + y^2 = 4$, where $\vec{F} = 3xy\hat{\imath} - y\hat{\jmath} + 2z\hat{k}$.

Sol: Given $\vec{F} = 3xy\hat{\imath} - y\hat{\jmath} + 2z\hat{k}$

$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$$

Consider, $\vec{F}.d\vec{r} = \left[3xy\hat{\imath} - y\hat{\jmath} + 2z\hat{k}\right].\left[dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right]$

$$\vec{F}.d\vec{r} = (3xy)dx + (-y)dy + (2z)dz \quad \text{---} \quad \textbf{(1)}$$

In $C : x^2 + y^2 = 4 \quad , z = 0$

$$x^2 + y^2 = 2^2 \qquad (x^2 + y^2 = r^2)$$

Put $x = rcos\theta \quad ; \quad y = rsin\theta \quad ; \quad z = 0$

$$x = 2cos\theta \quad ; \quad y = 2sin\theta$$

$$dx = -2sin\theta d\theta \quad ; \quad dy = 2cos\theta d\theta \quad ; \quad \mathrm{dz} = 0$$

$$\theta : \theta = 0 \quad to \quad \theta = 2\pi$$

(1)$\Rightarrow \vec{F}.d\vec{r} = (3.2cos\theta.2sin\theta)(-2sin\theta d\theta) + (-2sin\theta).(2cos\theta d\theta)$

$$\vec{F}.d\vec{r} = (-24cos\theta sin^2\theta - 4sin\theta cos\theta)d\theta$$

$$\int_C \vec{F}.d\vec{r} = \int_{\theta=0}^{2\pi} (-24sin^2\theta cos\theta - 4sin\theta cos\theta)d\theta$$

$$\int_C \vec{F}.d\vec{r} = -24\left[\frac{sin^3\theta}{3}\right]_0^{2\pi} - 4\left[\frac{sin^2\theta}{2}\right]_0^{2\pi} \qquad \left\{\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right\}$$

$$\int_C \vec{F}.d\vec{r} = -8(0-0) - 2(0-0) \qquad \{sin2\pi = 0 = sin0\}$$

$$\int_C \vec{F}.d\vec{r} = 0.$$

3. Find the total work done by a force $\vec{F} = 2xy\hat{\imath} - 4z\hat{\jmath} + 5x\hat{k}$ along the curve $x = t^2$, y=(2t+1) , $z = t^3$ from the point $t = 1$ to $t = 2$.

Sol: Given $\vec{F} = 2xy\hat{\imath} - 4z\hat{\jmath} + 5x\hat{k}$

$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$$

Consider, $\vec{F}.d\vec{r} = [2xy\hat{\imath} - 4z\hat{\jmath} + 5x\hat{k}].[dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}]$

$$\vec{F}.d\vec{r} = 2xydx - 4zdy + 5xdz \quad \text{--- } \mathbf{(1)}$$

In $C : x = t^2$; $\quad y = 2t + 1 \quad ; \quad z = t^3$

$dx = 2tdt$; $\quad dy = 2dt \quad ; \quad dz = 3t^2dt$

$t : t = 1 \quad to \quad t = 2$

(1)⇒ $\vec{F}.d\vec{r} = (2t^2(2t+1))2tdt - 4(t^3).2dt + (5t^2)3t^2dt$

$$\vec{F}.d\vec{r} = (8t^4 + 4t^3 - 8t^3 + 15t^4)dt$$

$\vec{F}.d\vec{r} = (23t^4 - 4t^3)dt$

The work done by a force $= \int_C \vec{F}.d\vec{r}$

$$\int_C \vec{F}.d\vec{r} = \int_{t=1}^{2} (23t^4 - 4t^3)dt$$

$$= 23\left[\frac{t^5}{5}\right]_{t=1}^{2} - 4\left[\frac{t^4}{4}\right]_{t=1}^{2}$$

$$= \frac{23}{5}(2^5 - 1) - (2^4 - 1)$$

$$= \frac{23(31)}{5} - 15 = \frac{713-75}{5}$$

Work done , $\int_C \vec{F}.d\vec{r} = \frac{638}{5}$ units

Green's Theorem

Let $M(x, y)$ and $N(x, y)$ be two functions defined in region $'R'$ and the $xy - Plane$ with simple closed curve C has its boundary , then $\oint_C \boldsymbol{Mdx + Ndy} = \iint_R \left[\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right] \boldsymbol{dydx}$

Note:

1. Area $= \iint_R \boldsymbol{dydx} = \frac{1}{2}\int_C (\boldsymbol{xdy - ydx})$

Problems

1. Evaluate $\int_C (\boldsymbol{xy - x^2})\boldsymbol{dx + x^2ydy}$ where C is the closed curve bounded by $y = 0$, $x = 1$ and $y = x$.

Sol: *Green's Theorem:* $\oint_C \boldsymbol{Mdx + Ndy} = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] \boldsymbol{dydx}$ --- (1)

Given $\int_C (xy - x^2)dx + x^2ydy$

Here, $M = xy - x^2$ $\quad N = x^2y$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2xy$$

<u>In 'R'</u>

$x: x = 0 \ \ to \ \ x = 1$

$y: y = 0 \ \ to \ \ y = x$

$$(1) \Rightarrow \int_C (xy - x^2)dx + x^2ydy = \int_{x=0}^{1}\int_{y=0}^{x}(2xy - x)dydx$$

$$= \int_{x=0}^{1}\left[2x\left[\frac{y^2}{2}\right]_0^x - x[y]_0^x\right]dx$$

$$= \int_{x=0}^{1}[x(x^2 - 0) - x(x - 0)]dx$$

$$\int_C (xy - x^2)dx + x^2ydy = \int_{x=0}^{1} (x^3 - x^2)\, dx$$

$$= \left[\frac{x^4}{4}\right]_{x=0}^{1} - \left[\frac{x^3}{3}\right]_{x=0}^{1}$$

$$= \frac{1}{4} - \frac{1}{3} = \frac{3-4}{12}$$

$$\int_C (\boldsymbol{xy - x^2})\boldsymbol{dx} + \boldsymbol{x^2ydy} = \frac{\boldsymbol{-1}}{\boldsymbol{12}}$$

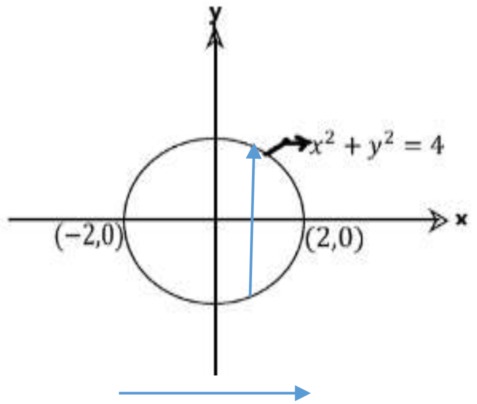
2. Use Green's theorem to evaluate $\int_C (\boldsymbol{x^2 + y^2})\boldsymbol{dx} + \boldsymbol{3x^2ydy}$ where C is the circle $x^2 + y^2 = 4$ traced in the positive sign.

Sol: *Green's Theorem:* $\oint_C \boldsymbol{Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dydx}$ --- (1)

Given $\int_C (x^2 + y^2)dx + 3x^2ydy$

Here, $M = x^2 + y^2$ $\quad N = 3x^2y$

$\boldsymbol{\frac{\partial M}{\partial y} = 2y}$ $\quad \boldsymbol{\frac{\partial N}{\partial x} = 6xy}$



<u>In 'R'</u>

$x: x = -2 \ \ to \ \ x = 2$

$y: y = -\sqrt{4 - x^2} \ \ to \ \ y = \sqrt{4 - x^2}$

$(1) \Rightarrow \int_C (x^2 + y^2)dx + 3x^2ydy = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (6xy - 2y)dydx$

$= \int_{x=-2}^{2} \left[6x\left[\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} - 2\left[\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right] dx$

$$= \int_{x=-2}^{2} \left[3x \left[\left[\sqrt{4-x^2}\right]^2 - \left[-\sqrt{4-x^2}\right]^2 \right] - \left[\left[\sqrt{4-x^2}\right]^2 - \left[-\sqrt{4-x^2}\right]^2 \right] \right] dx$$

$$\int_C (x^2+y^2)dx + 3x^2y dy = \int_{x=-2}^{2} [3x[(4-x^2)-(4-x^2)] - [(4-x^2)-(4-x^2)]]dx$$

$$\boldsymbol{\int_C (x^2+y^2)dx + 3x^2y dy = 0}.$$

Stoke’s Theorem

If S is a surface bounded by a simple closed curve C and if $\vec{F}$ is any continuously differentiable vector function then

$$\oint_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\hat{n}\,ds = \iint_S (\nabla X\vec{F}).\hat{n}\,ds$$

Problems

1.Verify Stoke’s theorem for $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Sol: By Stoke’s theorem,

$\oint_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\hat{n}\,ds$, C is the circle in the $xy - plane$ whose centre is the origin and radius equal to unity.

i.e., $x^2 + y^2 = 1$ and $\quad z = 0$

$$dz = 0$$

Put $x = cos\theta \; y = sin\theta \quad ; 0 \le \theta \le 2\pi$

LHS, $\oint_C \vec{F}.d\vec{r} = \oint_C ydx + zdy + xdz$

$$= \int_{\theta=0}^{2\pi} (sin\theta)(-sin\theta d\theta) + (0) + (0)$$

$$= -\int_{\theta=0}^{2\pi} sin^2\theta d\theta$$

$$= -\int_{\theta=0}^{2\pi} \frac{(1-cos2\theta)}{2} d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi} (1 - cos2\theta)d\theta$$

$$= -\frac{1}{2}\left\{[\theta]_0^{2\pi} - \left[\frac{sin2\theta}{2}\right]_0^{2\pi}\right\}$$

$$= -\frac{1}{2}\left\{[2\pi - 0] - \frac{1}{2}[sin4\pi - sin0]\right\}$$

$$\oint_C \vec{F}.d\vec{r} = -\frac{1}{2}(2\pi)$$

$$\oint_C \boldsymbol{\vec{F}.d\vec{r} = -\pi}$$

Now, $curl\vec{F} = \nabla X\vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(z)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(y)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y)\right]\hat{k}$$

$$= [0-1]\hat{\imath} - [1-0]\hat{\jmath} + [0-1]\hat{k}$$

$curl\vec{F} = -\hat{\imath} - \hat{\jmath} - \hat{k}$

$\hat{n}ds = dydz\hat{\imath} + dxdz\hat{\jmath} + dxdy\hat{k}$

$\hat{n}ds = 0.\hat{\imath} + 0.\hat{\jmath} + dxdy\hat{k}$ $(Since\ z = 0, dz = 0)$

$curl\vec{F}.\hat{n}ds = \left(-\hat{\imath} - \hat{\jmath} - \hat{k}\ \right).(0.\hat{\imath} + 0.\hat{\jmath} + dxdy\hat{k})$

$\boldsymbol{curl\vec{F}.\hat{n}ds = -dxdy}$

RHS, $\iint_S\ curl\vec{F}\,.\hat{n}\ ds = \iint_S\ -dxdy$

$$\iint_S curl\vec{F}\,.\hat{n}\ ds = -\iint_S dxdy$$

$\boldsymbol{\iint_S\ curl\vec{F}\,.\hat{n}\ ds = -\pi}$ $\left(\boldsymbol{Since} \iint_S\ dxdy = \boldsymbol{Area\ of\ circle}, x^2 + y^2 = 1 = \pi(1)^2 = \pi\right)$

LHS = RHS

2. Evaluate $\oint_C (xydx + xy^2dy)$ by Stoke's theorem where C is the square in $xy - plane$ with vertices $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$.

Sol: Given $\oint_C (xydx + xy^2dy)$

$$\oint_C \vec{F}.d\vec{r} = \oint_C (xydx + xy^2dy)$$

Here, $\vec{F} = xy\hat{\imath} + xy^2\hat{\jmath} + 0\hat{k}$

Now, $curl\vec{F} = \nabla X\vec{F}$

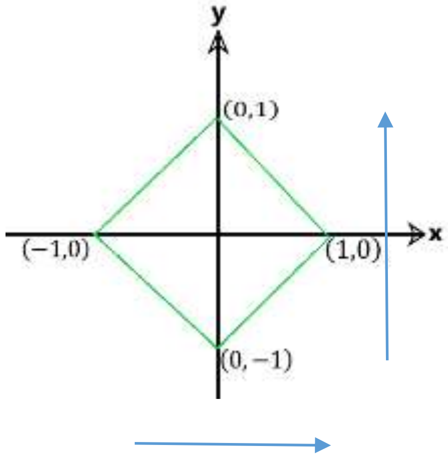
$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(xy^2)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(xy)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(xy)\right]\hat{k}$$

$$curl\vec{F} = [0-0]\hat{\imath} - [0-0]\hat{\jmath} + [y^2 - x]\hat{k}$$
$$\hat{n}ds = dydz\hat{\imath} + dxdz\hat{\jmath} + dxdy\hat{k}$$
$$curl\vec{F}.\hat{n}ds = \left(0.\hat{\imath} - 0.\hat{\jmath} + [y^2 - x]\hat{k}\right).(dydz\hat{\imath} + dxdz\hat{\jmath} + dxdy\hat{k})$$
$$\boldsymbol{curl\vec{F}.\hat{n}ds = \left[y^2 - x\right]dxdy}$$

Wkt , $\oint_C \boldsymbol{\vec{F}.d\vec{r}} = \iint_{\boldsymbol{R}} \boldsymbol{curl\vec{F}.\hat{n}\,ds}$

$$= \iint_R [y^2 - x]dxdy$$
$$= \int_{x=-1}^{1}\int_{y=-1}^{1}[y^2 - x]dydx$$
$$= \int_{x=-1}^{1}\left\{\left[\frac{y^3}{3}\right]_{-1}^{1} - x[y]_{-1}^{1}\right\}dx$$
$$= \int_{x=-1}^{1}\left[\frac{1}{3}[1-(-1)] - x[1-(-1)]\right]dx$$



<u>In 'R'</u>
$x: x = -1 \;\; to \;\; x = 1$
$y: y = -1 \;\; to \;\; y = 1$

$$\oint_C \vec{F}.d\vec{r} = \int_{x=-1}^{1} \left[\frac{2}{3} - 2x\right] dx$$

$$\oint_C \vec{F}.d\vec{r} = \frac{2}{3}[x]_{-1}^{1} - 2\left[\frac{x^2}{2}\right]_{-1}^{1}$$

$$\oint_C \vec{F}.d\vec{r} = \frac{2}{3}[1-(-1)] - [(1)^2-(-1)^2]$$

$$\oint_C \vec{F}.d\vec{r} = \frac{2}{3}[2] - [0]$$

$$\oint_C \boldsymbol{\vec{F}.d\vec{r}} = \boldsymbol{\frac{4}{3}}$$

Problem on flux

If $\vec{F} = 2xy\hat{\imath} + yz^2\hat{\jmath} + xz\hat{k}$ and S is the rectangular parallelepiped bounded by $x = 0, \ y = 0, z = 0, x = 2, y = 1, z = 3$.Find the flux across S.

Sol: Here$\vec{F} = 2xy\hat{\imath} + yz^2\hat{\jmath} + xz\hat{k}$

Now, $div\vec{F} = \nabla.\vec{F}$

$$= \left[\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right].\left[2xy\hat{\imath} + yz^2\hat{\jmath} + xz\hat{k}\right]$$

$$div\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$$

$$\boldsymbol{div\vec{F} = 2y + z^2 + x}$$

Flux across S $= \iint_S \vec{F}.\hat{n}\, ds$

By divergence theorem, $\iint_S \vec{F}.\hat{n}\, ds = \iiint_V div\vec{F}\, dv$

$$\iint_S \vec{F}\,.\,\hat{n}\,ds = \int_{z=0}^{3}\int_{y=0}^{1}\int_{x=0}^{2} (\mathbf{2y} + \mathbf{z^2} + \mathbf{x})\,dx\,dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{1}\left[(2y+z^2)\,[x]_0^2 + \left[\frac{y^2}{2}\right]_0^2\right] dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{1}(4y + 2z^2 + 2)\,dy\,dz$$

$$= \int_{z=0}^{3}\left\{4\left[\frac{y^2}{2}\right]_0^1 + (2z^2+2)[y]_0^1\right\}dz$$

$$= \int_{z=0}^{3}\{2 + 2z^2 + 2\}dz$$

$$= \int_{z=0}^{3}\{2z^2 + 4\}dz$$

$$\iint_S \vec{F}\,.\,\hat{n}\;ds = 2\left[\frac{z^3}{3}\right]_0^3 + 4[z]_0^3$$

$$= 2(9-0) + 4(3-0)$$

$$\boldsymbol{\iint_S \vec{F}\,.\,\hat{n}\;ds = 30}$$

Module-3: Partial Differential Equations

- Formation of PDE's by elimination of arbitrary constants and functions.
- Solution of non-homogeneous PDE by direct integration.
- Homogeneous PDEs involving derivative with respect to one independent variable only.
- Solution of Lagrange's linear PDE.
- Derivation of one-dimensional heat equation and wave equation.

Self-study: Solution of one-dimensional heat equation and wave equation by the method of separation of variables.

Definition

An equation involving one or more partial derivatives of a function of two or more variables is called a ***Partial differential equation[PDE]***.

The *order* of a PDE is the order of the highest derivative and the *degree* of highest order derivative after clearing the equation of fractional powers.

A PDE is said to be linear if it is of first first degree in the dependent variable and its partial derivatives.

If each term of the PDE contains either the dependent variable or one of its partial derivatives, the PDE is said to be ***homogeneous***. Otherwise it is said to be a ***nonhomogeneous*** PDE.

Examples

1. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ $\quad [Order - 1, Degree - 1, Homogeneous\ PDE]$

2. $\frac{\partial^2 z}{\partial x \partial y} = xy$ $[Order - 2, Degree - 1, NonHomogeneous\ PDE]$

3. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$ $[Order - 2, Degree - 1, Homogeneous\ PDE]$

4. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ $[Order - 2, Degree - 1, Homogeneous\ PDE]$

5. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$ $[Order - 3, Degree - 1, NonHomogeneous\ PDE]$

6. $\frac{\partial^2 u}{\partial x^2} = x + y$ $[Order - 2, Degree - 1, NonHomogeneous\ PDE]$

7. $\frac{\partial^2 z}{\partial x^2} - 16z = 0$ $[Order - 2, Degree - 1, Homogeneous\ PDE]$

8. $\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial u}{\partial y} = 0$ $[Order - 2, Degree - 1, NonHomogeneous\ PDE]$

Formation of PDE by eliminating arbitrary constants and arbitrary functions

Note:

If $z = f(x, y)$, then

$$z_x = \frac{\partial z}{\partial x} = p$$

$$z_y = \frac{\partial z}{\partial y} = q$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = r$$

$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = s$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = t$$

Problems

Form the PDE by eliminating the arbitrary constant in the following

1. $z = (x + a)(y + b)$

Sol: Given $z = (x + a)(y + b)$ --- (1)

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{\partial z}{\partial x} = (y + b).1 \quad \Rightarrow \quad \boldsymbol{p = (y + b)}$$

Also, $\frac{\partial z}{\partial y} = (x + a).1 \quad \Rightarrow \quad \boldsymbol{q = (x + a)}$

$(1) \Rightarrow z = q.p$

$\boldsymbol{z = pq}$, is the required PDE

Form the PDE by eliminating arbitrary functions in the following

1. $z = f(x^2 + y^2)$

Sol: Given $z = f(x^2 + y^2)$ --- (1)

Differentiating (1) partially w.r.t 'x' and 'y' , we have

$\frac{\partial z}{\partial x} = p = f'(x^2 + y^2).2x$

$\frac{\partial z}{\partial y} = q = f'(x^2 + y^2).2y$

$\frac{p}{x} = 2f'(x^2 + y^2)$ --- (2)

$\frac{q}{y} = 2f'(x^2 + y^2)$ --- (3)

From (2) and (3) , we have

$\frac{p}{x} = \frac{q}{y}$

$py = qx$

$\boldsymbol{py - qx = 0}$, is the required PDE.

Solution of Non Homogeneous PDE by Direct Integration

1. Solve $\frac{\partial^2 u}{\partial x^2} = x + y$

Sol: Given $\frac{\partial^2 u}{\partial x^2} = x + y$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = x + y$$

Integrate w.r.t $'x'$

$$\frac{\partial u}{\partial x} = \int x\,dx + y\int 1\,dx$$

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + yx + f(y)$$

Again integrating w.r.t $'x'$

$$u = \frac{1}{2}\int x^2\,dx + y\int x\,dx + f(y)\int 1\,dx$$

$$u = \frac{1}{2}\left[\frac{x^3}{3}\right] + y\left[\frac{x^2}{2}\right] + f(y).x + g(y)$$

$\boldsymbol{u = \frac{x^3}{6} + \frac{x^2 y}{2} + xf(y) + g(y)}$ **,** is the required solution.

Solution of Homogeneous PDE involving derivatives with respect to one independent variable only

1. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when $x = 0, z = e^y$ and $\frac{\partial z}{\partial x} = 1.$

Sol: Given $\frac{\partial^2 z}{\partial x^2} + z = 0$

$$(D^2 + 1)z = 0$$

A.E : $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm\sqrt{-1}$$

$$\boldsymbol{m = \pm i}$$

$$z_c = e^{0.x}[c_1 cosx + c_2 sinx]$$

The G.S is

$z = z_c$

$\boldsymbol{z = c_1 cosx + c_2 sinx}$

The solution for PDE is

$\boldsymbol{z = f(y)cosx + g(y)sinx}$ **--- (1)**

By data, When $x = 0, z = e^y$

$(1) \Rightarrow \; e^y = f(y)\cos(0) + g(y)\sin(0)$ $[\sin(0) = 0, \cos(0) = 1]$

$e^y = f(y)$

$\boldsymbol{f(y) = e^y}$

Differentiate (1) w.r.t ‘x’

$\boldsymbol{\frac{\partial z}{\partial x} = -f(y)sinx + g(y)cosx}$ **--- (2)**

when $\frac{\partial z}{\partial x} = 1$ and $x = 0$

$(2) \Rightarrow \quad 1 = -f(y)\sin(0) + g(y)\cos(0)$ $\quad [\sin(0) = 0, \cos(0) = 1]$

$1 = g(y)$

$\boldsymbol{g(y) = 1}$

$\therefore \quad (1) \Rightarrow \quad z = e^y \cos x + 1.\sin x$

$\boldsymbol{z = e^y \cos x + \sin x}$, is the solution.

Solution of the Lagrange's linear PDE

Lagrange's linear PDE is of the form $Pp + Qq = R$. Let us consider two equations: $u(x,y,z) = c_1$, $v(x,y,z) = c_2$ where $c_1$and c_2 are constants.

By the rule of cross multiplication we have,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Above equation is regarded as a system of simultaneous equations in three variables and relations $u(x,y,z) = c_1$, $v(x,y,z) = c_2$ satisfy these equations.

Thus $\varphi(u,v) = 0$ is a general solution of Lagrange's linear PDE.

Problems

1. Solve $yzp + xzq = xy$

Sol: Given $(yz)p + (xz)q = xy$

$$\langle Pp + Qq = R \rangle$$

Here $P = yz$ $\quad$ Q $= xz$ $\quad$ R $= xy$

Auxilary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

Comparing,

$$\frac{dx}{yz} = \frac{dy}{xz}$$

$$\frac{dy}{xz} = \frac{dz}{xy}$$

$$\frac{dx}{y} = \frac{dy}{x}$$

Integrating

$$\int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

$$x^2 = y^2 + 2c_1$$

$$x^2 - y^2 = k_1$$

$$\frac{dy}{z} = \frac{dz}{y}$$

Integrating

$$\int y\,dy = \int z\,dz$$

$$\frac{y^2}{2} = \frac{z^2}{2} + c_2$$

$$y^2 = z^2 + 2c_2$$

$$y^2 - z^2 = k_2$$

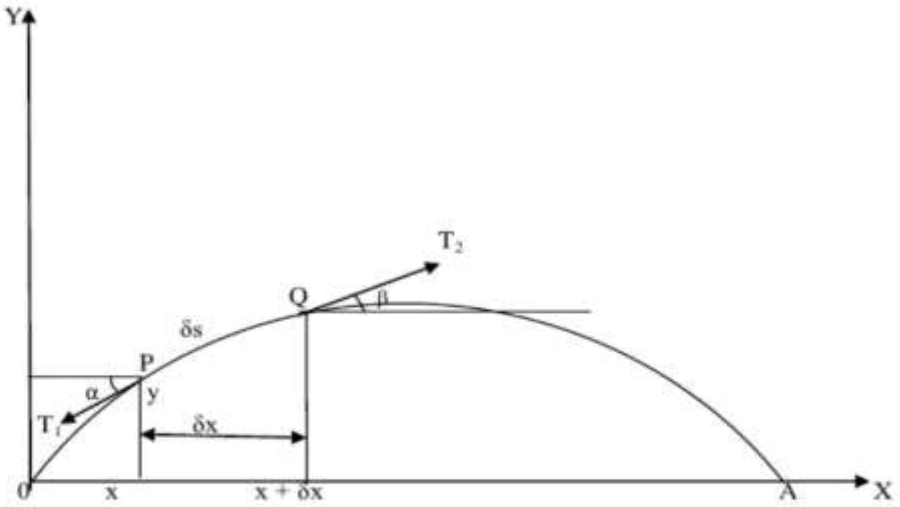
Thus the solution is, $\varphi(u, v) = 0$

$$\boldsymbol{\varphi\left((x^2 - y^2), (y^2 - z^2)\right) = 0.}$$

Derivation of One dimensional Wave Equation $[u_{tt} = c^2 u_{xx}]$

Consider a flexible string tightly stretched between two fixed points at a distance ‘l’ apart. Let $'\rho'$ be the mass per unit length. We assume

1. The tension T of the string is same throughout.
2. The effect of gravity can be ignored due to large tension ‘T’.
3. The motion of string is in small transverse vibration.



Let us consider the forces acting on a small element PQ of length δx.

Let T_1 and T_2 be the tensions at the points P and Q.

$\therefore T_1 cos\alpha = T_2 cos\beta = T$ [Since there is no motion in horizontal direction]

$T_1 cos\alpha = T$, $T_2 cos\beta = T$

$cos\alpha = \frac{T}{T_1}$, $cos\beta = \frac{T}{T_2}$

$\frac{1}{cos\alpha} = \frac{T_1}{T}$, $\frac{1}{cos\beta} = \frac{T_2}{T}$--- (1)

Vertical component of tension are $-T_1 sin\alpha$ and $T_2 sin\beta$, where the $-ve\ \ sign$ is used because T_1 is directed downwards.

Thus, Resultant force $F = T_2 sin\beta - T_1 sin\alpha$

Applying Newton's second law of motion,

$F = mass * acceleraion$

$$T_2 sin\beta - T_1 sin\alpha = \rho\delta x * \frac{\partial^2 u}{\partial t^2}$$ [density=mass/length]

Divide throughout by T

$$\frac{T_2}{T} sin\beta - \frac{T_1}{T} sin\alpha = \frac{\rho\delta x}{T} * \frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{cos\beta} . sin\beta - \frac{1}{cos\alpha} . sin\alpha = \frac{\rho\delta x}{T} * \frac{\partial^2 u}{\partial t^2}$$ [From (1)]

$$tan\beta - tan\alpha = \frac{\rho}{T} . \delta x . \frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho}{T} . \delta x . \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T} . \frac{\partial^2 u}{\partial t^2}$$

$$\lim_{x\to 0}\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}=\lim_{x\to 0}\frac{\rho}{T}.\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\rho}{T}.\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\rho}{T}.\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{T}{\rho}.\frac{\partial^2 u}{\partial x^2}$$

$$\boldsymbol{\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}} \qquad \left[c^2=\frac{T}{\rho}\right]$$

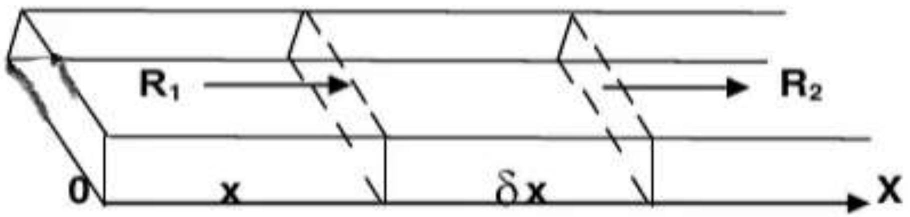
OR

$$\boldsymbol{u_{tt}=c^2u_{xx}}$$

Derivation of One dimensional Heat Equation $[u_t = c^2 u_{xx}]$

Consider a homogeneous bar of uniform cross section A. Suppose that the sides are covered with a material impervious to heat so that streamlines of heat-flow are all parallel and perpendicular to area. Take one end of the bar as the origin and the direction of flow as the positive x-axis.

Let $u = u(x,t)$ be the temperature of the slab at a distance x from the origin.Consider an element of slab between the plane $PQRS$ and ABCD at a distance x and $x + \delta x$ from the end O .Let ρ be the density,s the specific heat and k the thermal conductivity.



Let δu be the change in temperature in a slab of thickness δx of the bar.

The mass of the element $= A\rho\delta x$

The quantity of heat stored in the slab element $= A\rho s\delta x\delta u$

∴ Rate of increase of heat in this slab element is R $= (A\rho s\delta x)\frac{\partial u}{\partial t}$

Let R_1 be the rate of inflow of heat and R_2 is the outflow of heat.

We have, $R_1 = -kA\left[\frac{\partial u}{\partial x}\right]_x$, $R_2 = -kA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x}$

The negative sign is due to decrease in temperature as increase in t

Thus, $R = R_1 - R_2$

$$(A\rho s\delta x)\frac{\partial u}{\partial t} = -kA\left[\frac{\partial u}{\partial x}\right]_x + kA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x}$$

$$(A\rho s\delta x)\frac{\partial u}{\partial t} = kA\left[\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_x\right]$$ [Divide by A]

$$\rho s\frac{\partial u}{\partial t} = k\frac{\left[\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_x\right]}{\delta x}$$

$$\lim_{x\to 0}\rho s\frac{\partial u}{\partial t} = \lim_{x\to 0}k\frac{\left[\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_x\right]}{\delta x}$$

$$\rho s \frac{\partial u}{\partial t} = k \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial t} = \frac{k}{\rho s}\frac{\partial^2 u}{\partial x^2}$$

$$\boldsymbol{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \qquad \left[c^2 = \frac{k}{\rho s}\right]$$

OR

$$\boldsymbol{u_t = c^2 u_{xx}}$$

Module-4: Numerical Methods-1

- Solution of polynomial and transcendental equations: Regula-Falsi and Newton-Raphson methods (only formulae).
- Finite differences, Interpolation using Newton’s forward and backward difference formulae.
- Newton’s divided difference formula and Lagrange’s interpolation formula.
- Numerical integration: Simpson's (1/3)rd and (3/8)th rules(without proof)-Problems.

Self-study: Bisection method, Lagrange’s inverse Interpolation, Weddle's rule.

Numerical Methods

Numerical method provide various technique to find approximate solution to different problem using simple operation.

Numerical Solution of Polynomial and Transcendental Equations

Given an equation $f(x) = 0$ it is generally not possible to find roots $'x'$ such that $f(x)$ becomes zero exactly. We discuss two numerical methods for the solution of algebraic and transcendental equation.

Equation involving algebraic quantity like $x, x^2, x^3, \ldots$ are called ***algebraic equation***.

Eg: $x^3 - 3x - 4 = 0$, $x^4 + x^3 = 80$

Equation involving non algebraic quantity like $e^x, logx, sinx, tanx, \ldots$ are called ***transcendental equation***.

Eg: $xe^x - 2 = 0$, $xlogx - 1.2 = 0$, $tanx = 2x$

Numerical methods are often a repetitive nature. This consist repeated execution of the same process where at each step to result of the previous step is used. This is known as iterative process.

We discuss two numerical methods

1. Regula-Falsi method
2. Newton-Raphson method

Regula-Falsi method (or) Method of False position

Formula:

$$x_n = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

Problems

1. Using the method of false position find the real root correct to 3 decimal places of the equation $x^3 + 5x - 11 = 0$.

Sol: Given $f(x) = x^3 + 5x - 11$

$f(0) = -11$

$f(1) = -5 < 0$

$f(2) = 7 > 0$

The root lies between $(1,2)$.

Wkt,

$$x_n = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

Step-1: a = 1 b = 2

$f(a) = -5$ $f(b) = 7$

$$x_1 = \frac{1.(7) - 2.(-5)}{(7) - (-5)}$$

$$x_1 = \frac{7+10}{7+5}$$

$$\boldsymbol{x_1 = 1.417}$$

$f(1.417) = -1.070 < 0$

$f(2) = 7 > 0$

The root lies between (1.417,2).

Step-2: a = 1.417 b = 2

$f(a) = -1.070$ $f(b) = 7$

$$x_2 = \frac{(1.417).(7)-2.(-1.070)}{(7)-(-1.070)}$$

$$\boldsymbol{x_2 = 1.494}$$

$f(1.494) = -0.195 < 0$

$f(2) = 7 > 0$

The root lies between (1.417,2).

Step-3: a = 1.494 b = 2

$f(a) = -0.195$ $f(b) = 7$

$$x_3 = \frac{(1.494)(7)-2(-0.195)}{(7)-(-0.195)}$$

$$\boldsymbol{x_3 = 1.508}$$

$f(1.508) = -0.031 < 0$

$f(2) = 7 > 0$

The root lies between (1.508,2).

Step-4: a = 1.508 b = 2

$f(a) = -0.031$ $f(b) = 7$

$$x_4 = \frac{(1.508)(7)-2(-0.031)}{(7)-(-0.031)}$$

$$\boldsymbol{x_4 = 1.510}$$

$f(1.510) = -0.007 < 0$

$f(2) = 7 > 0$

The root lies between (1.510,2).

Step-5: a = 1.510 b = 2

$f(a) = -0.007$ $f(b) = 7$

$$x_5 = \frac{(1.510)(7)-2(-0.007)}{(7)-(-0.007)}$$

$$\boldsymbol{x_5 = 1.510}$$

∴ The real root is 1.510

Newton-Raphson Method

Formula:

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad ; f'(x_n) \neq 0 \quad , n = 0,1,2,3, \dots$$

$$n = 0 \, , x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \, ; f'(x_0) \neq 0$$

$$n = 1 \, , x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \, ; f'(x_1) \neq 0$$ and so on.

Problems

1. Use NR method to find the real root of the equation $x^3 - 3x - 5 = 0$, correct to 3 decimal places.

Sol: Given $f(x) = x^3 - 3x - 5$

$f(0) = -5$

$f(1) = -7$

$f(2) = -3$

$f(3) = 13$

The root lies between (2,3).

Since $f(2)$ lies nearer to 0.

Let $x_0 = 2$

We know that $\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

Here $f(x) = x^3 - 3x - 5$

$f'(x) = 3x^2 - 3$

Step 1: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_1 = 2 - \frac{f(2)}{f'(2)}$$

$$x_1 = 2 - \frac{(-3)}{(9)}$$

$$\boldsymbol{x_1 = 2.333}$$

Step 2: $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.333 - \frac{f(2.333)}{f'(2.333)}$$

$$x_2 = 2.333 - \frac{(0.699)}{(13.329)}$$

$$\boldsymbol{x_2 = 2.281}$$

Step 3: $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.281 - \frac{f(2.281)}{f'(2.281)}$$

$$x_3 = 2.281 - \frac{(0.025)}{(12.609)}$$

$$\boldsymbol{x_3 = 2.279}$$

Step 4: $$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 2.279 - \frac{f(2.279)}{f'(2.279)}$$

$$x_4 = 2.279 - \frac{(0)}{(12.582)}$$

$$\boldsymbol{x_4 = 2.279}$$

∴ The real root is 2.279.

FINITE DIFFERENCES

Newton's Forward interpolation formula (NFIF)

The value of $y = f(x)$ at $x = x_0 + rh$ is approximately given by

$$y = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots + \frac{r(r-1)(r-2)\ldots[r-(n-1)]}{n!}\Delta^n y_0$$

Where, $'r'$ is any real number , $r = \frac{x-x_0}{h}$; h is step length.

Newton's Backward interpolation formula (NBIF)

The value of $y = f(x)$ at $x = x_n + rh$ is approximately given by

$$y = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots + \frac{r(r+1)(r+2)\ldots[r+(n-1)]}{n!}\nabla^n y_n$$

Where, $'r'$ is any real number , $r = \frac{x-x_n}{h}$; h is step length.

1. The population of a town is given by the following data

Year	1971	1981	1991	2001	2011
Population (in thousand)	19.96	39.65	58.81	77.18	94.58

Using appropriate interpolation formula calculate the increase in the population from the year 1975 to 2005.

Sol: a) To find $\boldsymbol{f}(\mathbf{1975}) \Rightarrow x = 1975$

Here $h = 10$, $x_0 = 1971$

$$r = \frac{x - x_0}{h}$$

$$r = \frac{1975 - 1971}{10}$$

$$\boldsymbol{r = 0.4}$$

x	y	I D	II D	III D	IV D
$x_0 = 1971$	$y_0 = 19.96$				
		$\Delta y_0 = 19.69$			
1981	39.65		$\Delta^2 y_0 = -0.53$		
				$\Delta^3 y_0 = -0.26$	
		19.16			
			-0.79		$\Delta^4 y_0 = 0.08 = \nabla^4 y_n$
1991	58.81				
				$\nabla^3 y_n = -0.18$	
		18.37			
			$\nabla^2 y_n = -0.97$		
2001	77.18				
		$\nabla y_n = 17.40$			
$x_n = 2011$	$y_n = 94.58$				

By NFIF,

$$y = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$y = 19.96 + (0.4)(19.69) + \frac{(0.4)(0.4-1)}{2}(-0.53) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(-0.26) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(0.08)$$

$$y = 27.8797$$

Thus, $\boldsymbol{f(1975) = 27.8797}$

b) To find f(2005) $\Rightarrow x = 2005$

Here $h = 0.05$, $x_n = 2011$

$$r = \frac{x - x_n}{h}$$

$$r = \frac{2005-2011}{10}$$

$$\boldsymbol{r = -0.6}$$

By NBIF,

$$y = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n$$

$$y = 94.58 + (-0.6)(17.40) + \frac{(-0.6)(-0.6+1)}{2}(-0.97) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-0.18) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(0.08)$$

$$y = 84.2638$$

Thus, $\boldsymbol{f(2015) = 84.2638}$

The increase in population from the year 1975 to 2005= $\boldsymbol{f(1975) - f(2005)}$

$$= \mathbf{84.2638 - 27.8797}$$

$$= \mathbf{56.3841} \text{ (in thousands)}$$

Interpolation formula for unequal intervals

Newton's Divided Difference Formula or Newton's general interpolation formula

If $f(x_0), f(x_1), f(x_2)$, ..., $f(x_n)$ be a set of values of an unknown function $f(x)$ corresponding to the values x_0, x_1, x_2,…, x_n at unequal intervals, then

$$y = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \cdots$$

$$+(x - x_0)(x - x_1) \dots (x - x_{n-1})f(x_0, x_1, \dots, x_n)$$

Here $f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{(x_1-x_0)}$; $f(x_0, x_1, x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{(x_2-x_0)}$;

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{(x_3-x_0)}$$

Problems

1. Use NDDF to find $f(43)$, given that

x	40	42	44	45
$f(x)$	43833	46568	49431	50912

Sol:

x	$f(x)$	I DD	II DD	III DD
$x_0=40$	$f(x_0)=43833$			
		$f(x_0,x_1)=\frac{46568-43833}{42-40}=1367.5$		
$x_1=42$	$f(x_1)=46568$		$f(x_0,x_1,x_2)=\frac{1431.5-1367.5}{44-40}=16$	
		$f(x_1,x_2)=\frac{49431-46568}{44-42}=1431.5$		$f(x_0,x_1,x_2,x_3)=\frac{16.5-16}{45-40}=0.1$
$x_2=44$	$f(x_2)=49431$		$f(x_1,x_2,x_3)=\frac{1481-1431.5}{45-42}=16.5$	
		$f(x_2,x_3)=\frac{50912-49431}{45-44}=1481$		
$x_3=45$	$f(x_3)=50912$			

By NDDF,

$$y = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

$$y = 43833 + (43 - 40)(1367.5) + (43 - 40)(43 - 42)(16) + (43 - 40)(43 - 42)(43 - 44)(0.1)$$

$$\boldsymbol{y = 47983.2}$$

Thus, $\boldsymbol{f(43) = 47983.2}$

Lagrange's formula for interpolation

If $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ be a set of values of an unknown function $f(x)$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ not necessarily at equal intervals, then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1 + \cdots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\ldots(x_n-x_{n-1})} y_n$$

Lagrange's inverse interpolation formula for $x = f(y)$ is

$$x = f(y) = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)} x_1 + \cdots + \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)\ldots(y_n-y_{n-1})} x_n$$

Problems

1. Apply Lagrange's interpolation formula to find $y(11)$ from the following data

x	2	5	8	14
y	94.8	87.9	81.3	68.7

Sol: Given $x_0 = 2 \quad x_1 = 5 \quad x_2 = 8 \quad x_3 = 14$

$y_0 = 94.8 \quad y_1 = 87.9 \quad y_2 = 81.3 \quad y_3 = 68.7$

To find $y(11)$ $\Rightarrow x = 11$

Wkt

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = \frac{(11-5)(11-8)(11-14)}{(2-5)(2-8)(2-14)}(94.8) + \frac{(11-2)(11-8)(11-14)}{(5-2)(5-8)(5-14)}(87.9) + \frac{(11-2)(11-5)(11-14)}{(8-2)(8-5)(8-14)}(81.3) + \frac{(11-2)(11-5)(11-8)}{(14-2)(14-5)(14-8)}(68.7)$$

$y = 74.925$

$\boldsymbol{y(11) = 74.925}$

2. Fit an interpolating polynomial for the following data

x	0	1	2	5
y	2	3	12	147

Sol: Given $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 5$

$y_0 = 2 \quad y_1 = 3 \quad y_2 = 12 \quad y_3 = 147$

Wkt

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Wkt $(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$

$$y = \frac{(x^3 - 8x^2 + 17x - 10)}{(-10)}(2) + \frac{(x^3 - 7x^2 + 10x)}{(4)}(3) + \frac{(x^3 - 6x^2 + 5x)}{(-6)}(12) + \frac{(x^3 - 3x^2 + 2x)}{(60)}(147)$$

$$y = \frac{-(x^3 - 8x^2 + 17x - 10)}{5} + \frac{(3x^3 - 21x^2 + 30x)}{4} - \frac{(2x^3 - 12x^2 + 10x)}{1} + \frac{(147x^3 - 441x^2 + 294x)}{60}$$

$$y = \frac{-12(x^3 - 8x^2 + 17x - 10) + 15(3x^3 - 21x^2 + 30x) - 60(2x^3 - 12x^2 + 10x) + (147x^3 - 441x^2 + 294x)}{60}$$

$$y = \frac{-12x^3 + 96x^2 - 204x + 120 + 45x^3 - 315x^2 + 450x - 120x^3 + 720x^2 - 600x + 147x^3 - 441x^2 + 294x}{60}$$

$$y = \frac{60x^3 + 60x^2 - 60x + 120}{60}$$

$\boldsymbol{y = x^3 + x^2 - x + 2}$ **,** is the required polynomial.

Numerical Integration

The process of obtaining approximate value of the definite integration $I = \int_a^b y\,dx$ without actually integrating function but only using the value of $'y'$ at some point of x is equally placed over $[a, b]$.

Simpson's $\frac{1}{3}^{rd}$ rule

Formula:

$$\int_a^b y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})]$$

Where, $h = \frac{b-a}{n}$

Note: *To apply* $\frac{1}{3}^{rd}$ *rule n must be multiple of 2.*

Problems

1. Evaluate $\int_0^6 3x^2\, dx$ dividing the interval [0,6] in 6 equal parts (7 ordinate) by applying Simpson's $\frac{1}{3}^{rd}$ rule.

Sol: Given a = 0 , b = 6 , y = $3x^2$

Now, $h = \frac{b-a}{n}$

$h = \frac{6-0}{6}$

$\boldsymbol{h = 1}$, n = 6

x	0	1	2	3	4	5	6
y = $3x^2$	0	3	12	27	48	75	108

Wkt,

$$\int_a^b y\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_0^6 3x^2\,dx = \frac{1}{3}[(0 + 108) + 4(3 + 27 + 75) + 2(12 + 48)]$$

$$\int_0^6 \mathbf{3x^2\,dx = 216}.$$

Simpson's $\frac{3^{th}}{8}$ rule

Formula:

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})\right]$$

Where, $h = \frac{b-a}{n}$

Note: *To apply* $\frac{3^{th}}{8}$ *rule n must be multiple of 3.*

Problems

1. Evaluate $\int_0^1 \frac{1}{(1+x)}\,dx$ by taking 7 ordinate using Simpson's $\frac{3}{8}^{th}$ rule. Hence deduce the value of log2.

Sol: Given a = 0 , b = 1 , y = $\frac{1}{(1+x)}$

Now, $h = \frac{b-a}{n}$, n = 6

$$h = \frac{1-0}{6}$$

$$\boldsymbol{h = \frac{1}{6}}$$

x	**0**	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	**1**
$y = \frac{1}{(1+x)}$	1	0.8571	0.75	0.6667	0.6	0.5455	0.5

Wkt,

$$\int_a^b y\,dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_0^1 \frac{1}{(1+x)}dx = \frac{3 * \frac{1}{6}}{8}[(1 + 0.5) + 3(0.8571 + 0.75 + 0.6 + 0.5455) + 2(0.6667)]$$

$$\boldsymbol{\int_0^1 \frac{1}{(1+x)}dx = 0.6932}$$

By integration,

$$\int_0^1 \frac{1}{1+x}dx = [log(1+x)]_0^1$$

$$0.6932 = [log2 - log1]$$

$$\boldsymbol{log2 = 0.6932}$$

Module-5: Numerical Methods-2

Numerical solution of ordinary differential equations of first order and first degree

- Taylor’s series method.
- Modified Euler’s method.
- Runge-Kutta method of fourth-order.
- Milne’s predictor-corrector formula.

Self-study Adam-Bashforth method.

Numerical solution of ordinary differential equations of first order and first degree

A ***numerical method*** can be used to get an accurate approximate solution to a differential equation. There are many programs and packages available for solving these differential equations. With today's computer, an accurate solution can be obtained rapidly. In this chapter we focus on basic numerical methods for solving initial value problems.

Analytical methods, when available, generally enable to find the value of y for all values of x. Numerical methods, on the other hand, lead to the values of y corresponding only to some finite set of values of x. That is the solution is obtained as a table of values, rather than as continuous function. Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution inevitably involves an error which should be small but may, if it is not controlled, swamp the true solution. Therefore we must be concerned with two aspects of numerical solutions of ODEs: both the method itself and its accuracy.

The most general form of an ODE of first order and first degree is given by

$$\frac{dy}{dx} = f(x,y) where\ y(x_0) = y_0$$

Let x by an independent variable and y be dependent variable .

Let us consider the differential equation $\frac{dy}{dx} = f(x,y) where\ y(x_0) = y_0$--(1)

If particular values are given to the constants then the resulting solution is called a particular solution.

To obtain a particular solution from the general solution (1), we must be given initial conditions so that the constants can be determined. If all the initial conditions are specified at the same value of x then the problem is termed as initial value problem. If the conditions are specified at more than one value of x, then the problem is termed as boundary value problem.

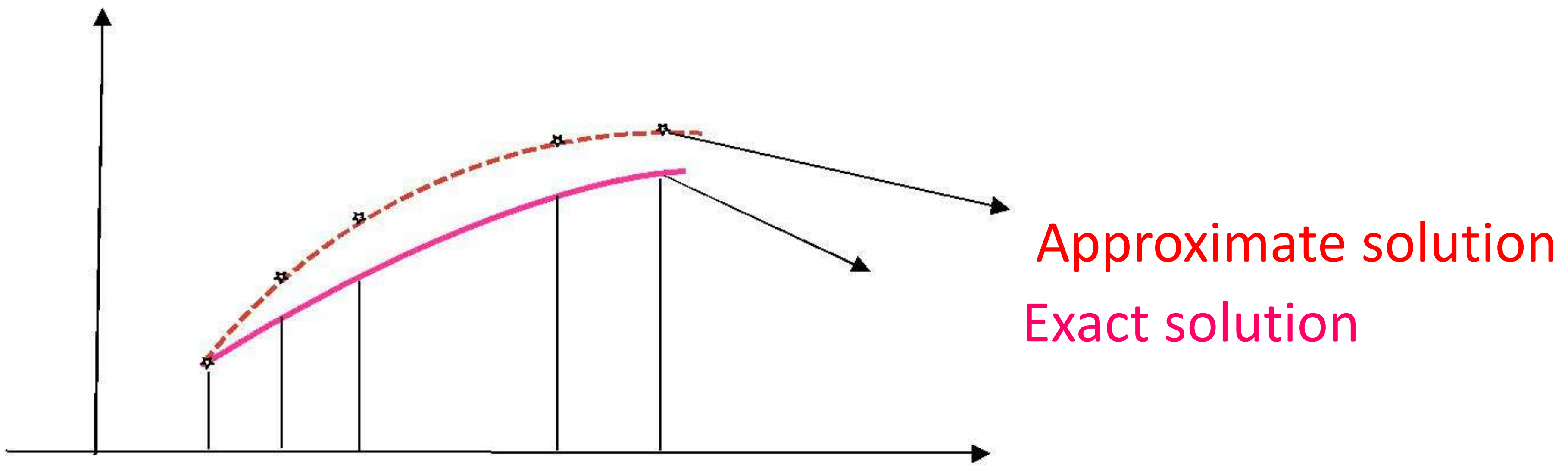
Though there are many analytical methods for finding the solution of the equation of the form (1), there exist large number of ODE’s whose solution cannot be obtained by the known analytical methods. In such cases, we use numerical methods to get an approximate solution of a given differential equation under the prescribed conditions

Consider the first order differential equation $\frac{dy}{dx} = f(x, y)$

Let $y(x_0), y(x_1), y(x_2), y(x_3)\ldots\ldots y(x_m)$ be the solution values at the points $x_{0,}$ $x_{1,}$ $x_{2,}$ $x_{3,}$----- x_m

We wish to find the approximate values y_0 , y_1, ..., y_m to these solution values.

Let the initial condition be $y(x_0) = y_0$. Let the exact solution y(x) of the given differential equation be represented by a continuous curve. Divide the interval (Xo,Xm))on which the solution is derived into a finite number of equispaced subintervals.



For each x_i, the approximate values of the dependent variable y(x) are calculated using a suitable recursive formula. These values are y_0, y_1, ..., y_m and these are shown by points. Computation of these approximate values is known as Numerical solution of the Differential equation.

An **Initial Value Problem** (IVP) consists of a differential equation and a condition which the solution much satisfies (or several conditions referring to the same value of x if the differential equation is of higher order).

$\frac{dy}{dx} = f(x, y) where\, y(x_0) = y_0$ --- (1)

The following methods are used to solve the IVP (1).

1. Taylor's Series Method

2. Modified Euler's Method

3. Runge - Kutta Method

4. Milne's Method

Taylor's Series Method

Let y=f(x) be a solution of the equation Expanding it by Taylor's series about $x - x_0$ we get

$$f(x) = y_0 + \frac{(x - x_0)}{1!} {y_0}^1 + \frac{(x - x_0)^2}{2!} {y_0}^2 + \frac{(x - x_0)^3}{3!} {y_0}^3 + \cdots.$$

1.Using the Taylor's series method , find an approximate solution correct to four decimals at x=0.1 for the IVP $\frac{dy}{dx} = 2y + 3e^x, y(0) = 0$

Sol:

$y_1 = 2y + 3e^x$ $\qquad$ $y_1(0) = 0 + 3e^0 = 3$

Differentiating w.r.t *x* we get

$y_2 = 2y_1 + 3e^x$ $\qquad$ $y_2(0) = 2 * 3 + 3e^0 = 9$

$y_3 = 2y_2 + 3e^x$ $\quad$ $y_3(0) = 2*9 + 3e^0 = 21$

$y_4 = 2y_3 + 3e^x$ $\quad$ $y_4(0) = 2*21 + 3e^0 = 45$

Wkt,

$$f(x) = y_0 + \frac{(x-x_0)}{1!}{y_0}^1 + \frac{(x-x_0)^2}{2!}{y_0}^2 + \frac{(x-x_0)^3}{3!}{y_0}^3 + \cdots.$$

Where x =0.1 and $x_0 = 0$ we get

$$y_1 = f(0.1) = 0 + \frac{0.1}{1!}(3) + \frac{0.1^2}{2!}(9) + \frac{0.1^3}{3!}(21) + \frac{0.1^4}{4!}(45) + \cdots$$

y(0.1)=0.3487

Modified Euler's Method

Consider the IVP $\frac{dy}{dx} = f(x, y), y(0) = y_0$ --- (1)

To determine the solution of this problem at $x_n = x_0 + nh$ bu using Euler's method.

$$\boldsymbol{y_n^{(E)} = y_{n-1} + hf(x_{n-1}, y_{n-1})} \text{ ---(2)}$$

The expression (2) gives an approximate value of y at x_n. . To improve the approximation the following formula has been suggested

$$\boldsymbol{y_n = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(E)})\right]} \text{---(3)}$$

The method of computing y_n using (3) is known as Modified Euler's method.

The process of improving the approximation can be continued by obtaining replacing $y_n^{(1)}, y_n^{(2)}, y_n^{(3)}$ until the desired degree of accuracy is obtained.

First approximation $y_1^{(E)} = y_0 + hf(x_0, y_0)$

Second approximation $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(E)})\right]$

Third approximation $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$ and so on.

1.Using the modified Euler's method, solve the IVP $\frac{dy}{dx} = \frac{1}{x+y}, y(0) = 1$ at points x=0.5 and x=1 in steps of length h=0.5. Carry out two modifications at each step.

Sol:

$\frac{dy}{dx} = f(x, y) = \frac{1}{x+y} \quad x_0 = 0\,, y_0 = 1 \; taking \; h = 0.5$

$$y_1^{(E)} = y_0 + hf(x_0, y_0) = y_0 + h\frac{1}{x_0+y_0} = 1 + \frac{0.5}{0+1} = \mathbf{1.5}$$

First modification $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(E)})\right]$

$$= y_0 + \frac{h}{2}\left[\frac{1}{x_0+y_0} + \frac{1}{x_0+y_1^E}\right]$$

$$= 1 + \frac{0.5}{2}\left[\frac{1}{0+1} + \frac{1}{0.5+1.5}\right] = \mathbf{1.375}$$

Second modification $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$$= y_0 + \frac{h}{2}\left[\frac{1}{x_0+y_0} + \frac{1}{x_0+y_1^1}\right]$$

$$= 1 + \frac{0.5}{2}\left[\frac{1}{0+1} + \frac{1}{0.5+1.375}\right] = \mathbf{1.3833}$$

Next, to compute the solution $y_2 = y(1)$ *and let us consider* $x_0 = 0.5\,, y_0 = 1.3833$

$$y_1^{(E)} = y_0 + hf(x_0, y_0) = y_0 + h\frac{1}{x_0+y_0} = 1.3833 + \frac{0.5}{0.5+1.3833} = \mathbf{1.6488}$$

First modification $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(E)})\right]$

$$= y_0 + \frac{h}{2}\left[\frac{1}{x_0+y_0} + \frac{1}{x_0+y_1^E}\right]$$

$$= 1.3833 + \frac{0.5}{2}\left[\frac{1}{0.5+1.3833} + \frac{1}{1+1.6488}\right] = \mathbf{1.6104}$$

Second modification $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$$= y_0 + \frac{h}{2}\left[\frac{1}{x_0+y_0} + \frac{1}{x_0+y_1^1}\right]$$

$$= 1.3833 + \frac{0.5}{2}\left[\frac{1}{0.5+1.3833} + \frac{1}{1+1.6104}\right] = \mathbf{1.6118}$$

The required solution are y(0.5)=1.3833 and y(1)=1.6118

RUNGE- KUTTA METHOD (R-K METHOD)

Consider the IVP $\frac{dy}{dx} = f(x,y), y(0) = y_0$ ---- (1)

To determine the solution of this problem at $x_n = x_0 + nh$ by using this method, where h is step length

According to the Euler's method , the solution at x_1 $is\ y_0 + hf(x_0, y_0)$.

This can be rewritten as $y_1 = y_0 + K$

Where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\boldsymbol{y_1 = y_0 + \frac{k_1+2k_2+2k_3+k_4}{6}}$$ is an approximate solution for the equation (1) at x_1 , known as Runge-Kutta method of order four.

1.Using RK method of fourth order , to find y(0.2), given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ $and\ y(0) = 1$ $take\ h = 0.2$

Sol:

Given $\frac{dy}{dx} = f(x,y) = \frac{y-x}{y+x}$ $, x_0 = 0, y_0 = 1\ and\ h = 0.2$ then $x_1 = x_0 + h = 0.2$

$k_1 = hf(x_0, y_0)$

$= h\left(\frac{y_0-x_0}{y_0+x_0}\right)$

$=(0.2)\left(\frac{1-0}{1+0}\right)$

$\boldsymbol{k_1 = 0.2}$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2\left(\frac{1.1 - 0.1}{1.1 + 0.1}\right)$$

$$\boldsymbol{k_2 = 0.1667}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2\left(\frac{1.0834 - 0.1}{1.0834 + 0.1}\right)$$

$$\boldsymbol{k_3 = 0.1662}$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.2\left(\frac{1.1662 - 0.2}{1.1662 + 0.2}\right)$$

$$\boldsymbol{k_4 = 0.1414}$$

Wkt,

$$y_1 = y_0 + \frac{k_1+2k_2+2k_3+k_4}{6}$$

$$= 1 + \frac{(0.2+2(0.1667+0.1662)+0.1414)}{6}$$

$$y(0.2) = 1.1679$$

Milne's Method:

The method in which the construction of y_n involves the use of not only y_{n-1} but also predecessors are called multi step methods. In multi-step methods two formulas are used in conjunction with each other- one for predicting the value of y_n and the other for correcting the predicted value of y_n.

Consider the IVP $\frac{dy}{dx} = f(x, y) --- -(1)$

Let $y_0=y(x_0)$,$y_1=y(x_1)$,$y_2=y(x_2)$ and $y_3=y(x_3)$ be these known solutions.

Suppose we wish to determine the solution of equation (1) at the point $x_4 = x_3+h$.

Let us denote the required solution by $y_4=y(x_4)$.

First we predict the value of $y_4=y(x_4)$ by using Milne's predictor formula:

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \qquad \text{--- (2)}$$

$which\ can\ be\ computed\ with\ the\ help\ of\ the\ specied\ x_0, x_1, x_2\ , x_3 and y_0, y_1, y_2, y_3.$

Next we correct the value of y_4 by using the Milne's corrector formula:

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(p)}\right) \qquad \text{--- (3)}$$

$$where \quad f_4^{(p)} = f(x_4, y_4^{(p)})$$

If we wish to have more accurate approximation for y, we employ the process repeatedly

1. Use Milne's predictor –corrector method to find the value of y at x=0.8, given $\frac{dy}{dx} = x - y^2$
   Where y(0)=0 , y(0.2)=0.02 , y(0.4)=0.0795 , y(0.6)=0.1762. Apply corrector formula twice.

Sol:

$Here\ f(x,y) = x - y^2, h = 0.2$

x	Y	$f(x,y) = x - y^2$
$x_0=0$	$y_0=0$	$f_{0=0-0^2=0}$
$x_1=0.2$	$y_1=0.02$	$f_{1=0.2-0.02^2=0.1996}$
$x_2=0.4$	$y_2=0.0795$	$f_{2=0.4-0.0795^2=0.3937}$
$x_3=0.6$	$y_3=0.1762$	$f_{3=0.6-0.1762^2=0.5689}$

Now , Milne's Predictor formula yields the predicted value of y_4 as

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 0 + \frac{4*0.2}{3}(2(0.1996) - 0.3937 + 2(0.5689))$$

$$\boldsymbol{y_4^{(p)} = 0.3048}$$

$$f_4^{(p)} = f(x_4, y_4^{(p)}) = x_4 - {y_4^{(p)}}^2 = 0.8 - (0.30488)^2$$

$$\boldsymbol{f_4^{(p)} = 0.7070}$$

Now , the Milne's corrector formula gives a corrected value of y_4 as

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(p)}\right)$$

$$= 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.7070)$$

$\boldsymbol{y_4^{(c)} = 0.3046}$

To apply corrector formula second time we must use corrector as predictor and substitute in $f_4^{(p)}$

$$f_4^{(p)} = f\left(x_4, y_4^{(p)}\right) = x_4 - {y_4^{(p)}}^2 = 0.8 - (0.3046)^2$$

$$\boldsymbol{f_4^{(p)} = 0.7072}$$

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(p)}\right) = 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.7072)$$

$$\boldsymbol{y_4^{(c)} = 03046}$$

Thus, $\boldsymbol{y(0.8) = 03046}$

Thank You